When are multidegrees positive?

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Degree

Let $X \subset \mathbb{P}^m_k$ be a projective variety over a field k. There are two important integers that provide information about X: **dimension** d and **degree** e.

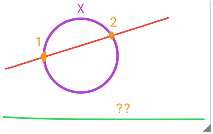
Geometrically, the degree is equal to the number of points in the intersection of X with a *generic* linear space L codimesion d.

Algebraically, the degree is $d!$ times the leading coefficient of the Hilbert polynomial of X.

The geometric definition need algebraically closed, as the next example will show.

Example

Let $X \subset \mathbb{P}^2_k$ be the zeros of the polynomial $x^2 + y^2 = z^2$, i.e., the circle. If $k = \mathbb{R}$ then there are generic lines with zero points in the intersection.



Algebraically, we have need to compute the Hilbert polynomial of the standard graded ring $R = k[x, y, z]/\langle x^2 + y^2 - z^2 \rangle$.

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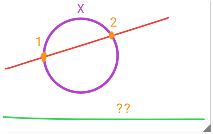


Algebraically, we have need to compute the Hilbert polynomial of the standard graded ring $R = k[x, y, z]/\langle x^2 + y^2 - z^2 \rangle$. We obtain

$$P_R(t) = \binom{t+2}{2} - \binom{t}{2} = 2t + 1 = \frac{\mathbf{2}}{1!} t^1 + 1$$

Extra assumptions

We are going to assume that our varieties are **reduced**. Furthermore, from the geometric intuition the degree is **additive** over the irreducible components of the variety. For these reasons we assume X is **irreducible** and on the algebra side we work with a standard* graded **domain** R which is a homogeneous quotient of $k[x_1, \cdots, x_m]$.

* each variable has degree 1.

Multidegrees

Let $X \subset \mathbb{P}^m \times \mathbb{P}^m \times \mathbb{P}^m$ be an irreducible variety over a field k. As a variety, X has some **dimension** d but now there is no single degree. Instead, we intersect with products of linear spaces whose codimensions add d.

Multidegrees

Let $X \subset \mathbb{P}^m \times \mathbb{P}^m \times \mathbb{P}^m$ be an irreducible variety over a field k. As a variety, X has some **dimension** d but now there is no single degree. Instead, we intersect with products of linear spaces whose codimensions add d.

Geometrically, for any $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{N}^3$ with $n_1 + n_2 + n_3 = d$ we define $\deg^{\mathbf{n}}(X)$ as the number of points in the intersection of X with a product $L_1 \times L_2 \times L_3$, where $L_i \subseteq \mathbb{P}^m$ is a general linear subspace of codimension n_i for each $1 \leq i \leq 3$.

Algebraically, we have a ring R that is a multihomogeneous quotient of $k[x_{ij} : 0 \leq i \leq m, 1 \leq j \leq 3]$ with standard grading $\deg(x_{ij}) = \mathbf{e}_j \in \mathbb{N}^3$. Then multidegrees appear in the multivariate Hilbert polynomial when expanded in as:

$$P_R(\mathbf{t}) = \sum_{\mathbf{n} \in \mathbb{N}^3} \deg^{\mathbf{n}}(R) \binom{t_1 + n_1}{n_1} \binom{t_2 + n_2}{n_2} \binom{t_3 + n_3}{n_3}$$

Chow rings

The Chow ring of $\mathbb{P} = \mathbb{P}_k^{m_1} \times \cdots \times \mathbb{P}_k^{m_p}$ is given by

$$A^*(\mathbb{P}) = \frac{\mathbb{Z}[H_1, \ldots, H_p]}{\left(H_1^{m_1+1}, \ldots, H_p^{m_p+1}\right)}$$

where H_i represents the class of the inverse image of a hyperplane of $\mathbb{P}_k^{m_i}$ under the natural projection $\Pi_i : \mathbb{P} \to \mathbb{P}_k^{m_i}$.
The class of the cycle associated to X coincides with

$$[X] = \sum_{\substack{0 \le n_i \le m_i \\ |\mathbf{n}| = d}} \deg_{\mathbb{P}}^{\mathbf{n}}(X)\, H_1^{m_1-n_1} \cdots H_p^{m_p-n_p} \in A^*(\mathbb{P}).$$

Question

Which classes are representable by irreducible varieties?

Example

Consider the affine toric variety $Z_A \subset \mathbb{A}^6$ given by the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 0 \\ 2 & 4 & 2 & 1 & 1 & 5 \end{bmatrix}.$$

By definition Z_A is the closure of the image of the monomial map given by

$$(\mathbb{C}^*)^2 \longrightarrow \mathbb{A}^6$$
$$(t_1, t_2) \mapsto (t_1^1 t_2^2, t_1^2 t_2^4, t_1 t_2^2, t_1^2 t_2, t_1 t_2, t_2^5)$$

Using M2 we can compute its toric ideal. This ideal is a binomial prime ideal so that Z_A is irreducible. We can now compute its closure Y_A on $\left(\mathbb{P}^2\right)^3$ by homogenizing. Then $Y_A \subset \left(\mathbb{P}^2\right)^3$ is an irreducible subvariety. Using M2 to compute its multidegree we obtain (notice that dimension is two):

$$5H_1^2H_2^2 + 10H_1^2H_2H_3 + 3H_1^2H_3^3 + 12H_1H_2^2H_3 + 6H_1H_2H_3^2.$$

How to interpret

We got $5H_1^2H_2^2 + 10H_1^2H_2H_3 + 3H_1^2H_3^3 + 12H_1H_2^2H_3 + 6H_1H_2H_3^2$.

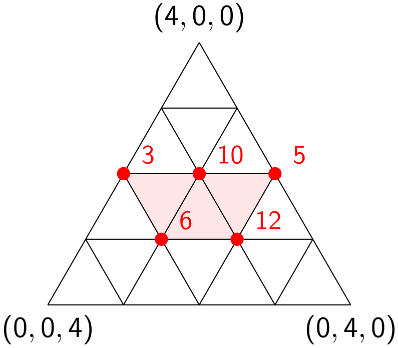
- The summand $5H_1^2H_2^2$ indicates that the *generic* intersection with H_3^2, that is, with something of the form $(\text{everything}, \text{everything}, \text{point})$ will have 5 points. So the multidegree of type $(0, 0, 2)$ is 5.
- The summand $10H_1^2H_2H_3$ indicates that a *generic* intersection with H_2H_3, that is, with something of the form $(\text{everything}, \text{line}, \text{line})$ will have 10 points. So the multidegree of type $(0, 1, 1)$ is 10.

Remark

There is a term $0H_2^2H_3^2$. In other words, the generic intersection with something of the form $(\text{point}, \text{everything}, \text{everything})$ is zero. The multidegree of type $(2, 0, 0)$ is zero.

Graphical representation

Above we were thinking of multidegrees as polynomials. It turns out that it is really helpful to have the following pictorial representation.



$$5H_1^2H_2^2 + 10H_1^2H_2H_3 + 3H_1^2H_3^3 + 12H_1H_2^2H_3 + 6H_1H_2H_3^2.$$

When are multidegrees positive?

Let k be an arbitrary field, $\mathbb{P} = \mathbb{P}_k^{m_1} \times \cdots \times \mathbb{P}_k^{m_p}$ be a multiprojective space over k, and $X \subseteq \mathbb{P}$ be a closed irreducible subscheme* of $\mathbb{P}$. *positivity only depends on the reduced structure

Theorem

Let $\mathbf{n} = (n_1, \ldots, n_p) \in \mathbb{N}^p$ be such that $n_1 + \cdots + n_p = \dim(X)$. Then, $\deg_{\mathbb{P}}^{\mathbf{n}}(X) > 0$ if and only if for each $\mathfrak{J} = \{j_1, \ldots, j_k\} \subseteq \{1, \ldots, p\}$ the following inequality holds

$$n_{j_1} + \cdots + n_{j_k} \leq \dim\left(\Pi_{\mathfrak{J}}(X)\right)$$

Furthermore, the function $r_X : 2^{[p]} \to \mathbb{Z}$ defined by $r_X(\mathfrak{J}) = dim\big(\Pi_{\mathfrak{J}}(X)\big)$ is submodular.

Theorem

When X is irreducible, the set

$$\mathrm{MSupp}_{\mathbb{P}}(X) = \left\{\mathbf{n} \in \mathbb{N}^p \mid \deg_{\mathbb{P}}^{\mathbf{n}}(X) > 0\right\},$$

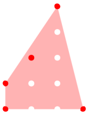
is a discrete polymatroid.

Newton polytopes

Let $p = \sum c_{\mathbf{a}} x^{\mathbf{a}} \in \mathbb{Z}[x_1, \cdots, x_p]$ be a polynomial. Then the **Newton polytope** $\mathrm{Newton}(p)$ is $\mathrm{conv}(\mathbf{a} \in \mathbb{Z}^p : c_{\mathbf{a}} \neq 0)$. The polynomial is said to have Saturated Newton Polytope (SNP) property if $\mathbf{a} \in \mathrm{Newton}(p) \iff c_{\mathbf{a}} \neq 0$ for all $\mathbf{a} \in \mathbb{Z}^p$.

Example

Let $p = 2x^3 - x^2y^4 + 7xy^2 + 6y + 3 \in k[x, y]$. Then p does not posses SNP.



Newton Polytopes, Multidegrees and Generalized permutohedra

We can restate our main theorem as

Theorem

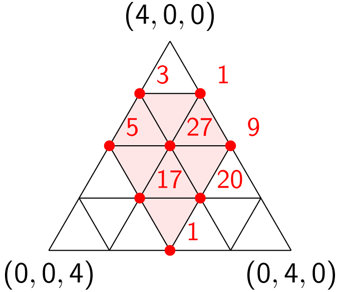
The Newton polytope of multidegrees (of irreducible varieties) is a generalized permutohedron and it has the SNP property.

Alternative definition

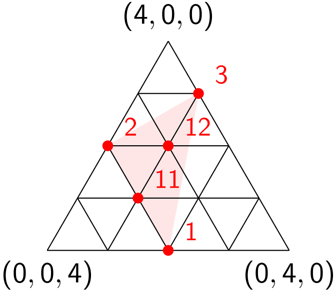
An important characterization is that a polytope is a generalized permutohedron if and only if all edges are parallel to edges of the standard simplex $\Delta_n := \mathrm{conv}(\mathbf{e}_i : i \in [n])$.

Representable classes

Which classes are representable?



Representable



Not Representable

Applications

1. Mixed multiplicities of ideals (new).
2. Schubert problems (Purbhoo-Sottile).
3. Multiprojective embedding of $\overline{M_{0,n}}$ (Cavalieri-Gillespie-Monin).
4. **Schubert polynomials** (Monical-Tockal-Yong, Fink-Mészáros-St.Dizier).
5. **Double Schubert polynomials** (C.-Cid Ruiz - Mohammadi - Montano).
6. Mixed volumes (Classical).

Schubert polynomials

The **Schubert polynomial** $\mathfrak{S}_\pi$ is the class of the Schubert variety $[Y_\pi]$ in the cohomology ring of the flag variety $\mathrm{Fl}(n)$ with the following presentation

$$R_n = k[x_1, \cdots, x_n]/\langle \text{symmetric polynomials w/o constant term.} \rangle$$

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Combinatorially they can be defined recursively as

$$\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$
$$\mathfrak{S}_{ws_i} = \frac{\mathfrak{S}_w - s_i \mathfrak{S}_w}{x_i - x_{i+1}}, \quad \ell(ws_i) < \ell(w),$$

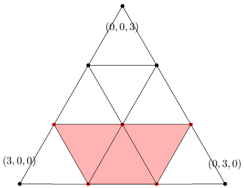
where $w_0 = n(n-1)\cdots 321$ is the *longest* element and $s_i := (i, i+1)$.
This is a well defined recursive process.

Example

$$\mathfrak{S}_{1432} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3.$$

Schubert polynomials

For the example before,
$\mathfrak{S}_{1432} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3$, we have



Schubert polynomials

Conjecture (Monical-Tokcan-Yong '17)

The Newton polytope of a Schubert polynomial is a generalized permutohedron and it is saturated. Furthermore, explicit inequalities were given.

Proved by Fink-Mészáros-St.Dizier in 2018.

We provide an alternative proof.

$\Longrightarrow$ Essentially same proof by Huh-Matherne-Mészáros-St.Dizier

Idea

Knutson-Miller proved that Schubert polynomials are the multidegrees of matrix schubert varieties.

Theorem (Castillo-Cid Ruiz-Mohammadi-Montano 21+)

The Newton polytope of a **double Schubert polynomial** is a generalized permutohedron and it is saturated.

Further Questions: Which classes are representable?

In the bigraded case Huh characterized representability up to a multiple.Gromov conjecture a general log concavity property. Burda showed it is not true.

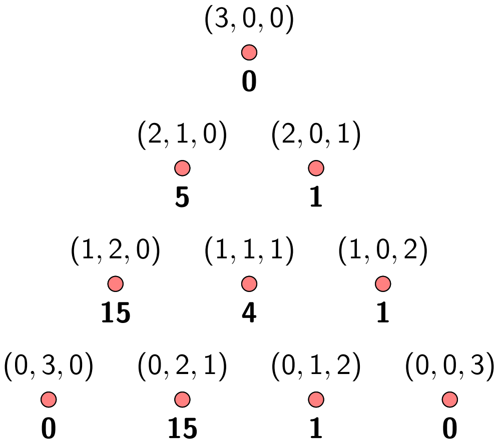


Figure: Burda's counterexample: $4^3 < 15 \times 5 \times 1$

Gracias!

Thank you for your attention!