The Polytope Algebra of Generalized Permutahedra

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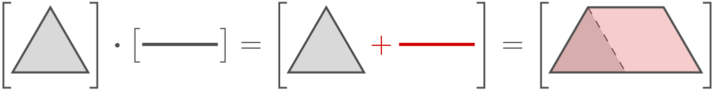
McMullen's polytope algebra

Let $\Pi(\mathbb{R}^d)$ be the abelian group generated by *classes* of polytopes $[\mathfrak{p}]$, with relations:

- $[\mathfrak{p} + \{t\}] = [\mathfrak{p}]$ for any *translation vector* $t \in \mathbb{R}^d$.
- $[\mathfrak{p}] + [\mathfrak{q}] = [\mathfrak{p} \cup \mathfrak{q}] + [\mathfrak{p} \cap \mathfrak{q}]$, whenever $\mathfrak{p} \cup \mathfrak{q}$ is a polytope.

Minkowski sum of polytopes defines a product on $\Pi(\mathbb{R}^d)$: $[\mathfrak{p}] \cdot [\mathfrak{q}] := [\mathfrak{p}+\mathfrak{q}]$.



$1 = [\cdot]$ $[\mathfrak{p}]^n = [n\mathfrak{p}]$

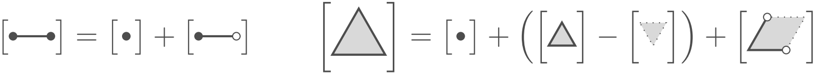
McMullen's polytope algebra

Theorem (McMullen '89)

$\Pi(\mathbb{R}^d)$ *is a graded* $\mathbb{R}$*-algebra:*

$$\Pi(\mathbb{R}^d) = \bigoplus_{r=0}^{d} \Pi_r(\mathbb{R}^d).$$

If $\mathfrak{p}$ *is* k*-dimensional:* $[\mathfrak{p}] = 1 + [\mathfrak{p}]_1 + \cdots + [\mathfrak{p}]_k.$



$$[\mathfrak{p}+\mathfrak{q}] = [\mathfrak{p}] \cdot [\mathfrak{q}] = 1 + [\mathfrak{p}]_1 + [\mathfrak{q}]_1 + \text{higher degree terms}$$

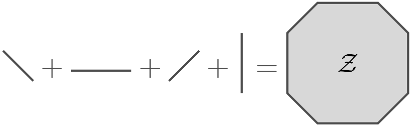
$$[n\mathfrak{p}] = [\mathfrak{p}]^n = 1 + n[\mathfrak{p}]_1 + \text{higher degree terms}$$

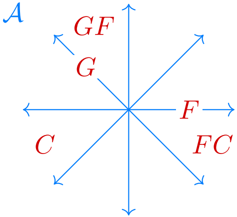
weighted Minkowski sums
$\updownarrow$
linear combinations in Π_1

$$\left[\sum_i \lambda_i \mathfrak{p}_i\right]_1 = \sum_i \lambda_i [\mathfrak{p}_i]_1$$

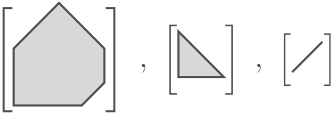
Zonotope subalgebra

Let $\mathcal{Z} \subseteq \mathbb{R}^d$ be a **zonotope**.





$$\Pi(\mathcal{Z}) = \bigoplus_{r=0}^{d} \Pi_r(\mathcal{Z}) = \text{subring of } \Pi(\mathbb{R}^d) \text{ generated by classes of } \textbf{deformations} \text{ of } \mathcal{Z}.$$



$\in \Pi(\mathcal{Z})$

- Only depends on the corresponding hyperplane arrangement $\mathcal{A}$.
- $\Pi(\mathcal{Z})$ is finite-dimensional.
- **McMullen '93:** If $\mathcal{Z}$ is **simple**, then $\dim_{\mathbb{R}}(\Pi_r(\mathcal{Z})) = h_r(\mathcal{Z})$.

How does $\Pi(\mathcal{Z})$ interact with the Tits (face) algebra of $\mathcal{A}$?

Zonotope subalgebra/module

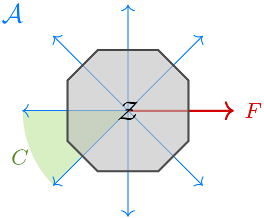
$\mathbb{R}\Sigma[\mathcal{A}]$ is the Tits algebra of $\mathcal{A}$.

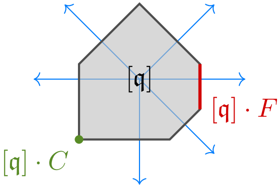
Theorem (B- '21)

$\Pi(\mathcal{Z})$ is a right $\mathbb{R}\Sigma[\mathcal{A}]$-module:

$$[\mathfrak{q}] \cdot F = [\textit{face of } \mathfrak{q} \textit{ maximized in the direction of } F]$$

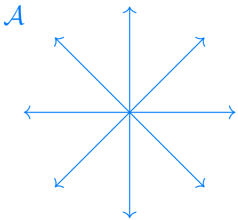
and each graded piece $\Pi_r(\mathcal{Z})$ is a submodule.

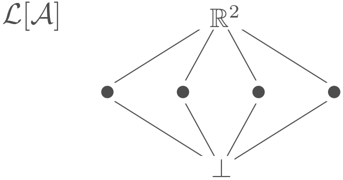




What can we say about the module structure of each graded piece $\Pi_r(\mathcal{Z})$?

FURTHER DECOMPOSITION (TECHNICAL SLIDE)





Saliola '06, Aguiar-Majahan '17:

Complete system of orthogonal idempotents $\{\mathtt{E}_{\mathrm{X}}\}_{\mathrm{X}\in\mathcal{L}[\mathcal{A}]}$ for $\mathbb{R}\Sigma[\mathcal{A}]$

This yields decompositions

$$\Pi_r(\mathcal{Z}) = \bigoplus_{\mathrm{X}\in\mathcal{L}[\mathcal{A}]} \Pi_r(\mathcal{Z}) \cdot \mathtt{E}_{\mathrm{X}}$$

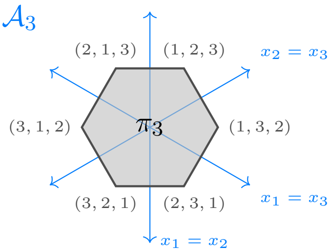
For each flat X:

$$\dim_{\mathbb{R}} \left(\Pi_r(\mathcal{Z}) \cdot \mathtt{E}_{\mathrm{X}}\right) = \sum_{\mathrm{Y}\geq\mathrm{X}} \mu(\mathrm{X},\mathrm{Y}) \dim_{\mathbb{R}} \left(\Pi_r(\mathcal{Z}_{\mathrm{Y}})\right)$$

$\rightsquigarrow$ in simplicial case:
refinement of h-numbers

Subalgebra of generalized permutahedra $\Pi(\pi_d)$

Braid arrangement $\mathcal{A}_d$ symmetric group $\mathfrak{S}_d$ permutahedron $\pi_d \subseteq \mathbb{R}^d$



Theorem (Björner '84, Brenti '94)

The h numbers of the W-permutahedron equal the W-Eulerian numbers.

$A_{\mathbf{d},\mathbf{r}} = \#$ permutations $\sigma \in \mathfrak{S}_{\mathbf{d}}$ with $\mathbf{r}$ **excedances** $i \in \mathrm{Exc}(\sigma)$ if $\sigma(i) > i$

$A_{3,0} = 1 \quad e$ $\quad A_{3,1} = 4 \quad (\mathbf{12}),(\mathbf{13}),(\mathbf{24}),(\mathbf{132})$ $\quad A_{3,2} = 1 \quad (\mathbf{123})$

Theorem (B- '21)

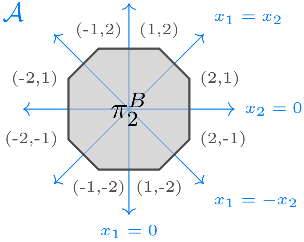
For any flat $\mathrm{X} \in \mathcal{L}[\mathcal{A}_d]$ *and* $\mathbf{r} = 0, 1, \ldots, d-1$,

$$\dim_{\mathbb{R}} \left(\Pi_{\mathbf{r}}(\pi_d) \cdot \mathtt{E}_{\mathrm{X}}\right) = \#\{\sigma \in \mathfrak{S}_d \,:\, \mathrm{exc}(\sigma) = \mathbf{r},\ \mathrm{fix}(\sigma) = \mathrm{X}\}.$$

Ardila-Sanchez, B-: $\Pi(\pi_I)$ is a Hopf monoid quotient of Aguiar-Ardila's **GP**

Subalgebra of type B generalized permutahedra $\Pi(\pi_d^B)$

Cox. arr. of type B $\mathcal{A}_d^{\pm}$ hyperoctahedral group $\mathfrak{B}_d$ type B perm. $\pi_d^B \subseteq \mathbb{R}^d$



Theorem (B- '21)

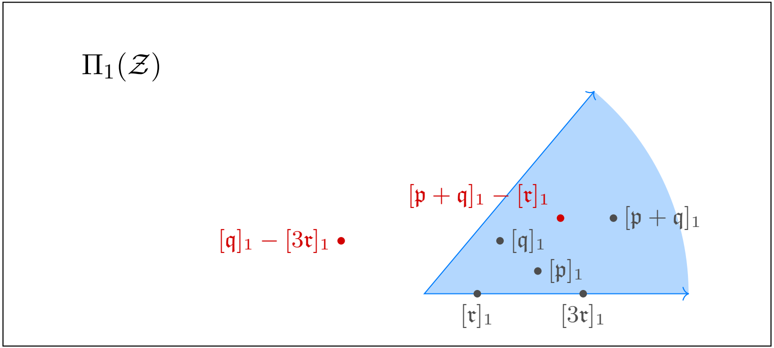
For any flat X $\in \mathcal{L}[\mathcal{A}_d^{\pm}]$ *and* $\mathbf{r} = 0, 1, \ldots, d$,

$$\dim_{\mathbb{R}} \left(\Pi_{\mathbf{r}}(\pi_d^B) \cdot \mathtt{E}_{\mathrm{X}} \right) = \#\{\sigma \in \mathfrak{B}_d \ : \ \mathrm{exc}_B(\sigma) = \mathbf{r}, \ \mathrm{fix}(\sigma) = \mathrm{X}\}.$$

exc_B is related to the *flag-excedance* statistic on signed permutations.

Deformation cone of $\mathcal{Z}$

Recall: weighted Minkowski sums $\leftrightarrow$ linear combinations on Π_1.



Question: Find a basis of $\Pi_1(\mathcal{Z})$ consisting of rays of $\mathrm{def}_t(\mathcal{Z})$.

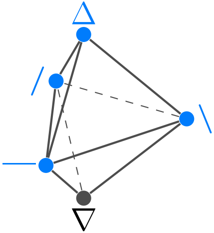
Proposition (B- '21)

Any such basis must contain at least $\dim_{\mathbb{R}}\left(\Pi_1(\mathcal{Z}) \cdot \mathtt{E}_{\perp}\right)$*-many polytopes of maximum dimension.*

Cone of generalized permutahedra

$$\dim_{\mathbb{R}}\left(\Pi_1(\pi_d)\cdot \mathtt{E}_{\perp}\right) = \#\{\sigma \in \mathfrak{S}_d \,:\, \mathrm{exc}(\sigma) = 1,\ \sigma \text{ cyclic}\} = \#\{(d\,d-1\,\ldots\,2\,\mathbf{1})\} = 1$$

A slice of $\mathrm{def}_t(\pi_3)$:



Theorem (Postnikov '09, Ardila-Benedetti-Doker '10)

Every generalized permutahedron in $\mathbb{R}^d$ can be written ***uniquely*** *as a signed Minkowski sum of the faces of the standard simplex $\Delta_{[d]}$.*

Constructive:

submodular functions $\leftrightarrow$ signed Minkowski sum of Δ_S

Cone of type B generalized permutahedra

Ardila-Castillo-Eur-Postnikov '20:
What is a *nice* type B analog of the last slide?

$$\dim_{\mathbb{R}} \left(\Pi_1(\pi_d^B) \cdot \mathtt{E}_\perp\right) = \#\{\sigma \in \mathfrak{B}_d \,:\, \mathrm{exc}_B(\sigma) = 1,\ \sigma\ \textit{cuspidal}\} = 2^{d-1}$$

- **Padrol-Pilaud-Ritter '20:** Shard polytopes. (14 full-dim. in $\mathbb{R}^3$)
- **B- '21:** slice the cross polytope

bisubmodular functions $\leftrightarrow$ signed Minkowski sum of Δ^0_S

Thank you!