Muttalib-Borodin Plane Partitions and the Hard Edge of Random Matrix Ensembles

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Aim

The main goal of this poster is to show how complex probabilistic behavior arises from the large scale behavior of *random* simple combinatorial objects.

In particular, one main result states that the asymptotic distribution of the peak of random trace-and-volume distributed plane partitions interpolates between two different extreme value statistics: Gumbel (universally the asymptotic max of uncorrelated rv's) and Tracy–Widom (universally the asymptotic max of correlated systems like eigenvalues of Hermitian random matrices).

An introduction to Muttalib–Borodin ensembles (MBEs)

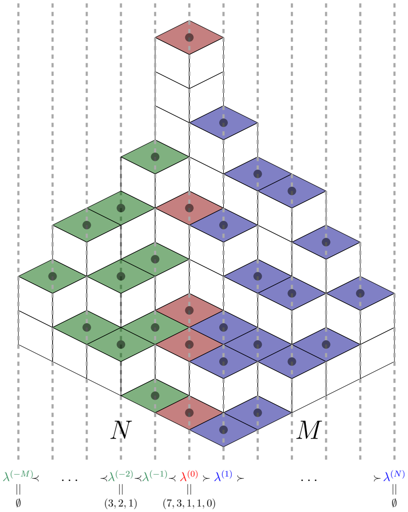
- probability measures on (usually ordered) N-tuples which have the form:

$$\mathbb{P}(x_1 \in dx_1, \ldots, x_N \in dx_N) \propto \prod_{i<j}(x_i - x_j)(x_i^\theta - x_j^\theta) \prod_i w(x_i) dx_i$$

  where $\theta > 0$ and $w(x) = e^{-V(x)}$ is a weight (potential)
- introduced by Muttalib [Mut95] as a simplified model for disordered conductors
- determinantal with "explicit" bi-orthogonal correlation kernel (Muttalib 1999)
- generalize unitarily invariant matrix models ($\theta = 1$)
- Borodin [Bor99] showed that for the Jacobi weight $w(x) = x^\alpha$ the correlation kernel is explicit; he also computed the hard-edge scaling limit (around 0) of the kernel in terms of a certain generalization of the Bessel kernel of Tracy–Widom (1994)
- Molag and Kuijlaar [KM19, Mol20] showed that Borodin's limiting kernel is universal ($1/\theta \in \mathbb{Z}$) for a wide variety of generic w's
- matrix models exist for specific w's (Forrester–Wang, Cheliotis)

Aim: explore discrete and continuous Jacobi-like MBEs ensembles and their edge scaling.

Plane partitions and discrete Muttalib–Borodin ensembles



- $\Lambda = (\Lambda_{i,j})$ an $M \times N$-based plane partition ($\Lambda_{i,j}$ = height of cubes above (i,j))
- consider prob. dist.

$$\mathbb{P}(\Lambda) \propto q^{\eta \cdot \text{left vol}} \left[a q^{\frac{\eta+\theta}{2}} \right]^{\text{center vol}} q^{\theta \cdot \text{right vol}}$$

- $0 \le a, q \le 1;\ \eta, \theta \ge 0;\ 1 \le M \le N \le \infty$
- Λ is a sequence of interlacing partitions:

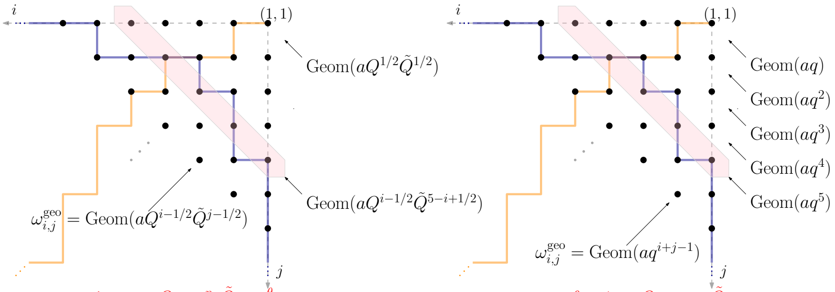
$$(\emptyset \prec \lambda^{(-M+1)} \prec \cdots \prec \lambda^{(0)} \succ \cdots \succ \lambda^{(N-1)} \succ \emptyset)$$

Thm. (B/Occ 2020; see also [FR05]) The middle (red) slice $l^{(0)}$ ($l_i^{(0)} = \lambda_i^{(0)} + M - i$) has a discrete MB dist. on M points:

$$\mathbb{P}(l^{(0)} = l) \propto \prod_{i<j} (Q^{l_j} - Q^{l_i})(\tilde{Q}^{l_j} - \tilde{Q}^{l_i})$$
$$\prod_i (a^2 Q \tilde{Q})^{\frac{l_i}{2}} (\tilde{Q}^{l_i+1}; \tilde{Q})_{N-M}$$

($(Q, \tilde{Q}) = (q^\eta, q^\theta)$); *all other slices* have similar MB dists. (discrete time MB process).

Last passage percolation (LPP) in a decaying infinite quadrant



generic case : $Q = q^\eta, \tilde{Q} = q^\theta$ case : $\eta = \theta = 1 \Leftrightarrow Q = q = \tilde{Q}$

$\omega^{\mathrm{geo}}_{i,j}$ are indep.; $X \sim \mathrm{Geom}(u)$ if $\mathbb{P}(X = k) = (1-u)u^k,\ k = 0, 1, \dots$

With $\pi \in$ {down-left paths from $(1,1) \to (\infty,\infty)$} (orange) and $\varpi \in$ down-right paths from $(\infty,1) \to (1,\infty)$} (blue), let:

$$L_1^{\mathrm{geo}} = \max_{\pi} \sum_{(i,j)\in\pi} \omega^{\mathrm{geo}}_{i,j}, \quad L_2^{\mathrm{geo}} = \max_{\varpi} \sum_{(i,j)\in\varpi} \omega^{\mathrm{geo}}_{i,j}$$

Thm. (B/Occ 2020) We have in distribution $L_1^{\mathrm{geo}} = L_2^{\mathrm{geo}} = \Lambda_{1,1} = l_1^{(0)}$ (the largest element of the discrete MBE above).

Pf: Burge+Knuth corresp. and Greene-type thms.

Edge scaling

Thm. (B/Occ 2020) Let $M = N = \infty$. In the $q = e^{-\epsilon} \to 1-, \epsilon \to 0+$ we have, for any $L \in \{L_1^{\mathrm{geo}}, L_2^{\mathrm{geo}}, \Lambda_{1,1}\}$:

- if $a = e^{-\alpha\epsilon} \to 1-$ then

$$\lim_{\epsilon\to 0+} \mathbb{P}\left(\epsilon L + \frac{\log(\epsilon\eta)}{\eta} + \frac{\log(\epsilon\theta)}{\theta} < s\right) = \det(1 - \tilde{K}_{he})_{L^2(s,\infty)}$$

where the RHS is a Fredholm determinant of the operator $\tilde{K}_{he}(x,y)$ with

$$\tilde{K}_{he}(x,y) = e^{-\frac{x}{2}-\frac{y}{2}} K_{he}(e^{-x}, e^{-y}) \text{ and}$$

$$K_{he}(x,y) = \frac{1}{\sqrt{xy}} \int_{\delta+i\mathbb{R}} \frac{d\zeta}{2\pi i} \int_{-\delta+i\mathbb{R}} \frac{d\omega}{2\pi i} \frac{F_{he}(\zeta)}{F_{he}(\omega)} \frac{x^\zeta}{y^\omega} \frac{1}{\zeta-\omega}, \quad F_{he}(\zeta) = \frac{\Gamma(\frac{\alpha}{2\eta} - \frac{\zeta}{\eta} + \frac{1}{2})}{\Gamma(\frac{\alpha}{2\theta} + \frac{\zeta}{\theta} + \frac{1}{2})}.$$

- if $0 < a < 1$ is fixed, then

$$\lim_{\epsilon\to 0+} \mathbb{P}\left(\frac{L - c_1\epsilon^{-1}}{c_2\epsilon^{-1/3}} < s\right) = F_{\mathrm{TW}}(s)$$

where F_{TW} is the Tracy–Widom GUE distribution and c_1, c_2 are explicit.

If $\epsilon = \frac{1}{R}$ then $L = \begin{cases} O(R\log R) \text{ with O(R) fluctuations,} & \text{if } a \to 1- \\ O(R) \text{ with } O(R^{1/3}) \text{ fluctuations,} & \text{if } 0 < a < 1 \text{ fixed} \end{cases}, \quad R \to \infty.$

From Gumbel to Tracy–Widom via exponential Bessel

In the case of *equi-distributed diagonals* $\eta = \theta = 1$ we have:

$$\tilde{K}_{he}(x, y) = e^{-\frac{x}{2} - \frac{y}{2}} K_{\alpha,\mathrm{Bessel}}(e^{-x}, e^{-y})$$

with $K_{\alpha,\mathrm{Bessel}}(x, y) = \int_0^1 J_\alpha(2\sqrt{ux}) J_\alpha(2\sqrt{uy}) du$ the continuous Bessel kernel.

Then [Joh08] our limiting distribution $F_\alpha(s) = \det(1 - \tilde{K}_{he})_{L^2(s,\infty)}$ interpolates between

- Gumbel ($\alpha = 0$), universal maximum of iid rv's
- and Tracy–Widom GUE ($\alpha \to \infty$ + rescaling), universal max of correlated rv's (e.g. eigenvalues of Hermitian random matrices)

Thm. (B/Occ 2020) For $\eta = \theta = 1$ and as $q \to 1-$, the rv L has:

- Gumbel fluctuations, if $a = 1$;
- Tracy–Widom fluctuations (on a different scale), if $0 < a < 1$ fixed;
- transitional (exponential) hard-edge Bessel fluctuations, if $a \to 1$ critically.

Fox-H kernel

Prop. (B/Occ 2020) We have

$$K_{he}(x,y) = \frac{1}{\sqrt{xy}} \int_0^1 f_{he}\left(\frac{1}{ux}\right) g_{he}(uy) \frac{du}{u}$$

where, with H the Fox H-function [Fox61]

$$(f_{he}(x), g_{he}(x)) = \left(H^{0,1}_{1,1}\left[x \,\middle|\, \begin{matrix} \left(\frac{\alpha}{2\eta} + \frac{1}{2}, \frac{1}{\eta}\right) \\ \left(\frac{\alpha}{2\theta} + \frac{1}{2}, \frac{1}{\theta}\right) \end{matrix} \right], H^{1,0}_{1,1}\left[x \,\middle|\, \begin{matrix} \left(\frac{\alpha}{2\eta} + \frac{1}{2}, \frac{1}{\eta}\right) \\ \left(\frac{\alpha}{2\theta} + \frac{1}{2}, \frac{1}{\theta}\right) \end{matrix} \right] \right).$$

(Can be written in terms of $J_{a,b}(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!\Gamma(a+bk)}$; $\eta = \theta$ or $\eta = 1, \theta \in \mathbb{Q}$ yields the Meijer-G kernel.)

Fox H-function

From Wikipedia, the free encyclopedia

In mathematics, the **Fox H-function** $H(x)$ is a generalization of the Meijer G-function and the Fox–Wright function introduced by Charles Fox (1961). It is defined by a Mellin–Barnes integral

$$H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (a_1,A_1) & (a_2,A_2) & \ldots & (a_p,A_p) \\ (b_1,B_1) & (b_2,B_2) & \ldots & (b_q,B_q) \end{matrix} \right] = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j + B_j s) \prod_{j=1}^n \Gamma(1 - a_j - A_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - B_j s) \prod_{j=n+1}^p \Gamma(a_j + A_j s)} z^{-s}\, ds$$

where L is a certain contour separating the poles of the two factors in the numerator. Compare to the Meijer G-function.

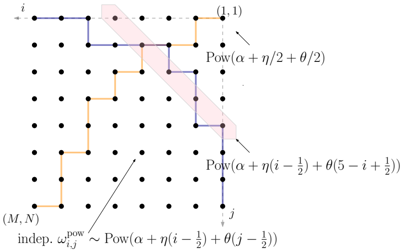
(`https://en.wikipedia.org/wiki/Fox_H-function`, with slightly modified conventions)

Continuous Jacobi-like MBE processes and last-passage percolation

Place $\omega_{i,j}^{\mathrm{pow}} \sim \mathrm{Pow}(\alpha + \eta(i - \frac{1}{2}) + \theta(j - \frac{1}{2}))$ for $(1,1) \leq (i,j) \leq (M,N)$ (indep.) and let

$$L_1^{\mathrm{pow}} = \min_{\pi} \prod_{(i,j)\in\pi} \omega_{i,j}^{\mathrm{pow}}, \quad L_2^{\mathrm{pow}} = \min_{\varpi} \prod_{(i,j)\in\varpi} \omega_{i,j}^{\mathrm{pow}}$$

over down-left paths π from $(1,1) \to (M,N)$ (orange) and down-right ϖ from $(M,1) \to (1,N)$ (blue).



$Y \sim \mathrm{Pow}(\beta)$ if $\mathbb{P}(Y \in dx) = \beta x^{\beta - 1} \chi_{[0,1]}$

Thm. (B/Occ 2020) We have $L_1^{\mathrm{pow}} = L_2^{\mathrm{pow}} = x_1$ in dist. with x_1 the smallest (hard-edge) point in the following M-point MB ensemble $\vec{x} = (0 < x_1 < \cdots < x_M < 1)$:

$$\mathbb{P}(\vec{x} \in d\vec{x}) \propto \prod_{1\leq i<j\leq M} (x_j^{\eta} - x_i^{\eta})(x_j^{\theta} - x_i^{\theta}) \prod_{1\leq i\leq M} x_i^{\alpha + \frac{\eta+\theta}{2} - 1}(1 - x_i^{\theta})^{N-M}.$$

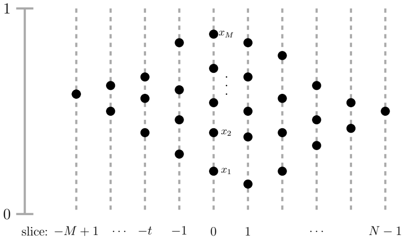
The Fox-H kernel at the hard-edge

Thm. (B/Occ 2020) With $L \in \{L_1^{\text{pow}}, L_2^{\text{pow}}, x_1\}$ we have the gap probability:

$$\lim_{M,N\to\infty} \mathbb{P}\left(L > \frac{r}{M^{\frac{1}{\eta}} N^{\frac{1}{\theta}}} \right) = \det(1 - K_{he})_{L^2(0,r)}$$

with K_{he} the Fox-H kernel.

A (discrete) time- and (continuous) space-extended version of this result holds with little modification.



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