Eulerian Representations for Real Reflection Groups

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BIG IDEA

TYPE A STORY:

TOPOLOGY	REPRESENTATION THEORY	COMBINATORICS
CONFIGURATION SPACE COHOMOLOGY	⟷ REPRESENTATION ISOMORPHISM	EULERIAN IDEMPOTENTS OF S_n

THEOREM (B, 2020): When W is of coincidental type

degrees form an arithmetic progression

This picture generalizes

TOPOLOGY	REPRESENTATION THEORY	COMBINATORICS
COHOMOLOGY OF "THICKENED" HYPERPLANE COMPLEMENT	⟷ REPRESENTATION ISOMORPHISM	EULERIAN IDEMPOTENTS OF W

AND for W a finite Coxeter group, there is a similar (more technical) statement

OUTLINE

I. MOTIVATING STORY: TYPE A

II. REFLECTION GROUP SETUP

III. COINCIDENTAL REFLECTION GROUPS

IV. COXETER GROUPS

I. Motivating Story: Type A

The story begins with descents...

For Permutations:

Let S_n be the symmetric group. For $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in S_n$,

$$\mathrm{Des}(\sigma) := \{ i \in [n-1] : \sigma_i > \sigma_{i+1} \} \qquad \mathrm{des}(\sigma) := \#\mathrm{Des}(\sigma)$$

Example:

If $\sigma = (1, 5, 3, 4, 2)$ $\qquad \mathrm{Des}(\sigma) = \{2, 4\}$ $\qquad \mathrm{des}(\sigma) = 2$

In Coxeter Theory:

Let (W, S) be a Coxeter system. For $w \in W$

$$\mathrm{Des}(w) := \{ s \in S : \ell(ws) < \ell(w) \} \qquad \mathrm{des}(w) := \#\mathrm{Des}(w)$$

THEOREM (Solomon, 1976)

For any Coxeter group W, there is a subalgebra $\mathrm{Sol}(W) \subset \mathbb{Q}W$ spanned by sums of elements with the same **DESCENT SET**.

Call this subalgebra **SOLOMON'S DESCENT ALGEBRA**

EXAMPLE: In S_3 ...

$\mathrm{Des}(\sigma)$	Element in $\mathrm{Sol}(S_3)$
$\emptyset \longleftrightarrow \emptyset$	$(1,2,3)$
$\{1\} \longleftrightarrow \{s_1\}$	$(2,1,3) + (3,1,2)$
$\{2\} \longleftrightarrow \{s_2\}$	$(1,3,2) + (2,3,1)$
$\{1,2\} \longleftrightarrow \{s_1, s_2\}$	$(3,2,1)$

DEFINITION (Garsia-Reutenauer, 1989):

The (Type A) **EULERIAN IDEMPOTENTS** are a family of orthogonal idempotents $e_k \in Sol(S_n)$ for $k = 0, 1, \ldots, n-1$, satisfying

$$\sum_{k=0}^{n-1} e_k t^{k+1} = \sum_{\sigma \in S_n} \binom{t-1+n-des(\sigma)}{n} \sigma$$

EXAMPLE: In S_3...

$$e_2 = \frac{1}{6}\left((1,2,3) + (2,1,3) + (3,1,2) + (1,3,2) + (2,3,1) + (3,2,1)\right)$$

$$e_1 = \frac{1}{2}\left((1,2,3) - (3,2,1)\right)$$

$$e_0 = \frac{1}{6}\left(2(1,2,3) - (2,1,3) - (3,1,2) - (1,3,2) - (2,3,1) + 2 \cdot (3,2,1)\right)$$

RECALL

- Irreducible representations of S_n are indexed by partitions of n
- For an irreducible rep S^{λ}, I will write λ

  e.g $S^{(2,1)} \longleftrightarrow \ydiagram{2,1}$

DEFINITION:

Each e_k generates an S_n representation called the k-th Eulerian representation defined by

$$E_k := \mathbb{Q} S_n \cdot e_k$$

FOR EXPERTS: $E_0 \cong \mathrm{Lie}_n$, the multilinear component of the free Lie algebra

EXAMPLE: For S_3, the Eulerian representations are

$$\left.\begin{aligned} E_2 &= \ydiagram{3} \\ E_1 &= \ydiagram{2,1} \oplus \ydiagram{1,1,1} \\ E_0 &= \ydiagram{2,1} \end{aligned}\right\} \text{reg. rep of } S_3$$

Mysteriously these representations arose in another context...

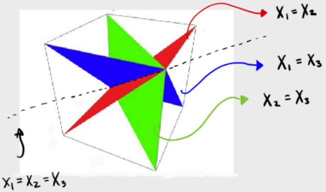
DEFINITION

The n^{th} ordered configuration space of $\mathbb{R}^d$ is

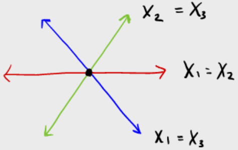
$$\mathrm{Conf}_n(\mathbb{R}^d) := \{(x_1, \overset{\in \mathbb{R}^d}{x_2}, \dots, x_n) \in \mathbb{R}^{dn} : x_i \neq x_j\}$$

EXAMPLE: When $d = 1$ and $n = 3$,

$$\mathrm{Conf}_3(\mathbb{R}) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_i \neq x_j\}$$



PROJECT along $x_1 = x_2 = x_3$



QUESTION: What is $H^*\mathrm{Conf}(\mathbb{R}^d)$?

THEOREM (Ar'nold $d=2$, F. Cohen, $d\geq 2$.)

cohomological degree $d-1$

For $d\geq 2$, $H^*\mathrm{Conf}_n(\mathbb{R}^d)\cong \mathbb{Z}\langle u_{ij}\rangle/I$

where $i,j\in[n]$ and I is generated by:

(1) $u_{ij}u_{k\ell}=(-1)^{d+1}u_{k\ell}u_{ij}$ (2) $u_{ij}^2=0$ (3) $u_{ij}=(-1)^{d+1}u_{ji}$

(4) $u_{ij}u_{jk}+u_{jk}u_{ki}+u_{ki}u_{ij}=0$

NOTE: * Presentation depends on parity of d

n pieces

* Cohomology is concentrated in degrees $0,\ d-1,\ 2(d-1),\ \ldots\ldots,\ (n-1)(d-1)$

QUESTION: What is $H^*\mathrm{Conf}(\mathbb{R}^d)$ as an S_n representation?

NOTE: S_n acts on $H^*\mathrm{Conf}_n(\mathbb{R}^d)$ by permuting indices

e.g. $\sigma\cdot u_{ij}=u_{\sigma(i)\sigma(j)}$

* When d is odd: $H^*\mathrm{Conf}_n(\mathbb{R}^d)\cong \mathbb{Q}[S_n]$

KEY CONNECTION:

There is an isomorphism of S_n-representations:

$$E^{(n)}_{n-1-k} \cong H^{(d-1)k}\,\mathrm{Conf}_n(\mathbb{R}^d, \mathbb{Q})$$

characters by Hanlon (1990)

characters by Sundaram-Welker (1997)

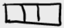
for every $k = 0, 1, 2, \ldots, n-1$, and $d \geq 3$ and odd

EXAMPLE:

For S_3 and $d = 3$ ...

for experts!

nbc-basis for $H^{2k}\,\mathrm{Conf}_3(\mathbb{R}^3)$:

$E_2 \cong$  $\cong H^0\,\mathrm{Conf}_3(\mathbb{R}^3) \longleftrightarrow 1$

$E_1 \cong$  $\oplus$  $\cong H^2\,\mathrm{Conf}_3(\mathbb{R}^3) \longleftrightarrow u_{12},\ u_{13},\ u_{23}$

$E_0 \cong$  $\cong H^4\,\mathrm{Conf}_3(\mathbb{R}^3) \longleftrightarrow u_{12}u_{23},\ u_{12}u_{13}$

In Summary:

Topology
Configuration Space Cohomology
(i)

Representation Theory
Representation Isomorphism
(iii)

Combinatorics
Eulerian Idempotents of S_n
(ii)

Question: Is this connection actually a more general phenomena?

Preview: Yes!

* very natural generalization for coincidental reflection groups
* more technical generalization for all finite Coxeter groups

II. REFLECTION GROUP SET UP

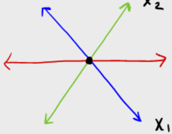
* Every finite Coxeter (reflection) group W defines a hyperplane arrangement $\mathcal{A}_W \subset V$ ← $\mathbb{R}$-vector space

$$\boxed{\text{reflections } s_\alpha \in W \; \underset{\text{FIXED SPACE}}{\overset{\text{REFLECT OVER}}{\rightleftarrows}} \; \text{hyperplanes } H_\alpha \in \mathcal{A}_W}$$

* The **RANK** of $\mathcal{A}_W$ is $\dim(V)$

EXAMPLE: $W = S_3$

$\mathcal{A}_{S_3} =$



$\text{rank}(\mathcal{A}_{S_3}) = 2$

RECALL:

TYPE A:

TOPOLOGY
CONFIGURATION SPACE COHOMOLOGY
(i)

REPRESENTATION THEORY
$\longleftrightarrow$
REPRESENTATION ISOMORPHISM
(iii)

COMBINATORICS
EULERIAN IDEMPOTENTS OF S_n
(ii)

GENERALIZATION: ?

(i) CONFIGURATION SPACE COHOMOLOGY

RECALL: $\mathrm{Conf}_n(\mathbb{R}^d) = \{(x_1, \ldots, x_n) \in \mathbb{R}^{dn} : x_i \neq x_j\}$

IDEA: Rephrase this using hyperplane arrangements,

e.g. $\mathcal{A}_{S_n} = \{H_{ij} \text{ for } 1 \leq i < j \leq n\}$ ← $\{x_i = x_j\} \subset \mathbb{R}^n$

DEFINITION: $\mathrm{Conf}_n(\mathbb{R}^d) = \mathbb{R}^n \otimes \mathbb{R}^d - \{\bigcup_{1 \leq i < j \leq n} H_{ij} \otimes \mathbb{R}^d\}$

and more generally, for any reflection group W

$$M_W^d := V \otimes \mathbb{R}^d - \{\bigcup_{H \in \mathcal{A}_W} H \otimes \mathbb{R}^d\}$$

for experts! This is a $\mathbb{Z}/2\mathbb{Z}$ orbit configuration space

EXAMPLE: For $W = B_n$,

$$M_{B_n}^d = \{(\underset{\in \mathbb{R}^d}{x_1}, \ldots, x_n) \in \mathbb{R}^{dn} : \underset{x_i \neq +/- x_j}{x_i \neq 0}\}$$

PROPERTIES OF M_W^d:

As in the case of $\mathrm{Conf}_n(\mathbb{R}^d)$...

* W acts on M_W^d and $H^* M_W^d$ carries a W representation

* Presentation of $H^* M_W^d$ depends on **PARITY** of d

  d **EVEN**: **ORLIK-SOLOMON ALGEBRA** (Orlik-Solomon, 1980)

  d **ODD**: associated graded **VARCHENKO-GELFAND RING** (Moseley, 2017)

  INTUITION: commutative version of Orlik-Solomon algebra

* $H^* M_W^d$ concentrated in degrees $0, 1(d-1), 2(d-1), \ldots, r(d-1)$

* When d is **ODD**: $H^* M_W^d \cong \mathbb{R}W$

(ii) Eulerian Idempotents

The **Eulerian Idempotents** have been extensively generalized:

* Bergeron-Bergeron (1992): Type B
* Bergeron-Bergeron-Howlett-Taylor (1992): finite Coxeter groups
* Saliola (2009): any central hyperplane arrangement
* Aguiar-Mahajan (2017): even more properties!

NOTE: Definitions are very technical

Let's accept they exist without definition for now...

"**DEFINITION:**"

$e_0, e_1, \ldots, e_r \in \mathrm{Sol}(W)$ are the Eulerian idempotents of W

$$E_k := \mathbb{R}We_k := k\text{-th Eulerian representation of } W$$

TYPE A:

TOPOLOGY
CONFIGURATION SPACE COHOMOLOGY

REPRESENTATION THEORY
REPRESENTATION ISOMORPHISM

COMBINATORICS
EULERIAN IDEMPOTENTS OF S_n

(i) (iii) (ii)

?

GENERALIZATION: $H^* M_W^d \longleftrightarrow e_k$

BIG Q: Does (iii) hold for arbitrary reflection groups?

i.e. what is the relationship between $H^{k(d-1)} M_W^d$ and E_{r-k}?

III. COINCIDENTAL REFLECTION GROUPS

DEFINITION: Every reflection group has an associated integer sequence $0 < d_1 \leq d_2 \leq \cdots \leq d_r$ called its **FUNDAMENTAL DEGREES**

for experts!

$\boxed{d_1, \ldots, d_r \doteq \text{polynomial degrees of the generators of } Sym(V^*)^W \cong \mathbb{R}[x_1 .. x_r]^W}$

EXAMPLE: The degrees of S_n are $d_1 = 2 \quad d_2 = 3 \quad \ldots \quad d_{n-1} = n$

DEFINITION: A finite reflection group is **coincidental** if its fundamental degrees form an arithmetic progression

e.g. S_n: $2 \xrightarrow{+1} 3 \xrightarrow{+1} 4 \xrightarrow{+1} \cdots \xrightarrow{+1} n$

B_n: $2 \xrightarrow{+2} 4 \xrightarrow{+2} 6 \xrightarrow{+2} 8 \cdots \xrightarrow{+2} 2n$

$I_2(m)$

H_3

WHY COINCIDENTAL? PART 1: EULERIAN SUBALGEBRAS

TYPES A AND B: There is a subalgebra $s(W)$ of $Sol(W)$ spanned by sums of elements with the same **DESCENT NUMBER**.

Garsia-Reutenauer (A)

Bergeron-Bergeron (B)

EXAMPLE: For S_3:

$Des(\sigma)$	Element in $Sol(S_3)$
ϕ	$(1,2,3)$
$\{1\}$	$(2,1,3) + (3,1,2)$
$\{2\}$	$(1,3,2) + (2,3,1)$
$\{1,2\}$	$(3,2,1)$

$des(\sigma)$	Element in $s(S_3)$
0	$(1,2,3)$
1	$(2,1,3) + (3,1,2) + (1,3,2) + (2,3,1)$
2	$(3,2,1)$

THEOREM (B, 2020): There is a subalgebra $s(W)$ of $Sol(W)$ spanned by elements with the same **DESCENT NUMBER** **IF AND ONLY IF** W is **COINCIDENTAL**.

The $e_k \in s(W)$ in this case and in fact generate $s(W)$.

WHY COINCIDENTAL? PART 2: PRODUCT FORMULAS

Recall the definition of the $e_k \in \mathrm{Sol}(W)$ are quite technical, with notable exceptions:

TYPE A: (Garsia-Reutenauer)

$$\sum_{k=0}^{n-1} e_k t^{k+1} = \sum_{\sigma \in S_n} \binom{t-1+n-\mathrm{des}(\sigma)}{n} \sigma$$

TYPE B: (Bergeron-Bergeron)

$$\sum_{k=0}^{n} e_k t^k = \sum_{w \in B_n} \binom{\frac{t-1}{2}+n-\mathrm{des}(w)}{n} w$$

THEOREM (B, 2020):

For W coincidental of rank r, the Eulerian idempotents satisfy

$$\sum_{K=0}^{r} e_K t^K = \frac{1}{|W|} \sum_{w \in W} \prod_{i=1}^{\mathrm{des}(w)} (t-d_i+1) \prod_{i=1}^{r-\mathrm{des}(w)} (t+d_i-1) \cdot w$$

WHY COINCIDENTAL? PART 3: TOPOLGY

THEOREM (B, 2020)

Suppose W is a finite coincidental group of rank r.
Then for every k such that $0 \le k \le r$;
when $d \ge 3$ and odd,

$$V \otimes \mathbb{R}^d - \{\bigcup_{H \in \mathcal{A}_W} H \otimes \mathbb{R}^d\}$$

$$E_{r-k} \cong H^{k(d-1)} M_W^d$$

W-representation isomorphism

THIS CONFIRMS:

TOPOLOGY
COHOMOLOGY OF "THICKENED" HYPERPLANE COMPLEMENT

REPRESENTATION THEORY
$\longleftrightarrow$
REPRESENTATION ISOMORPHISM

COMBINATORICS
EULERIAN IDEMPOTENTS OF W

TECHNIQUES:

* Following Barr (1968), define a "shuffle element"

$$\mathcal{B} \in \mathrm{Sol}(W)$$

* Prove that $\mathcal{B}$ acts semisimply on $\mathbb{Q}W$

* Show eigenspaces of $\mathcal{B}$ are the E_k

* Important tools:
  - Equivariant BHR-Theory (Reiner-Saliola-Welker, 2014)
  - Equivariant Goresky-MacPherson (Sundaram-Welker, 1997)

IV. COXETER GROUPS

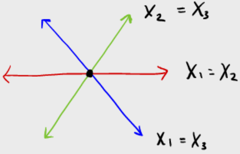
$\mathcal{A}_W$ defines a lattice:

DEFINITION:

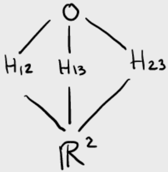
- $\mathcal{L}(\mathcal{A}_W)$ is the poset of intersection subspaces (e.g. flats) of $\mathcal{A}_W$.
- W acts on $\mathcal{L}(\mathcal{A}_W)$
- For $X \in \mathcal{L}(\mathcal{A}_W)$, $[X]$ is the W-orbit of X.

EXAMPLE: For S_3,

$\mathcal{A}_{S_3}$:



$\mathcal{L}(\mathcal{A}_{S_3})$:



$\mathcal{L}(\mathcal{A}_{S_3})/S_3$:

$[0]$

$|$

$[H_{12}]$

$|$

$[\mathbb{R}^2]$

THEOREM (B, 2020):

For any finite Coxeter group W, there is a family of orthogonal idempotents
$e_{[X]} \in Sol(W)$ for $[X] \in \mathcal{L}(\mathcal{A}_W)/W$
generating representations $E_{[X]} := \mathbb{R}W e_{[X]}$
such that for $d \geq 3$ and odd

$$\boxed{V \otimes \mathbb{R}^d - \{\bigcup_{H \in \mathcal{A}_W} H \otimes \mathbb{R}^d\}}$$

$$\bigoplus_{\substack{[X] \in \mathcal{L}(\mathcal{A}_W)/W \\ \operatorname{codim}(X) = k}} E_{[X]} \cong H^{(d-1)k} M_W^d$$

W-representation isomorphism

IN SUMMARY:

GOAL: generalize

TOPOLOGY	$\xleftrightarrow[\text{REPRESENTATION ISOMORPHISM}]{\text{REPRESENTATION THEORY}}$	COMBINATORICS
CONFIGURATION SPACE COHOMOLOGY		EULERIAN IDEMPOTENTS OF S_n

THEOREM (B, 2020):

FOR COINCIDENTAL GROUPS:

- Eulerian subalgebra of $\mathrm{Sol}(W) \Longleftrightarrow W$ coincidental
- Nice product formulas for e_k.
- The isomorphism: $E_{r-k} \cong H^{(d-1)k} M_W^d$ $d \geq 3$ odd

FOR ANY FINITE COXETER GROUP

- The isomorphism: $\displaystyle\bigoplus_{\substack{[X] \in \mathcal{L}(A_W)/W \\ \operatorname{codim}(X) = k}} E_{[X]} \cong H^{(d-1)k} M_W^d$ $d \geq 3$ odd

THANK YOU!

CONTACT ME!

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