Modules of the 0-Hecke algebra arising from standard permuted composition tableaux

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Definitions and notation

- The 0-Hecke algebra $H_n(0)$ of rank n is the algebra over $\mathbb{C}$ generated by $\pi_1, \dots, \pi_{n-1}$ subject to the relations:

$$\begin{aligned} \pi_i^2 &= \pi_i && \text{for } 1 \le i \le n-1 \\ \pi_i\pi_{i+1}\pi_i &= \pi_{i+1}\pi_i\pi_{i+1} && \text{for } 1 \le i \le n-2 \\ \pi_i\pi_j &= \pi_j\pi_i && \text{if } |i-j| \ge 2 \end{aligned}$$

- A *composition* α of n, denoted by $\alpha \models n$, is a finite ordered list of positive integers $(\alpha_1, \dots, \alpha_k)$ satisfying $\sum_{i=1}^k \alpha_i = n$. A *generalized composition* $\boldsymbol{\alpha}$ of n, denoted by $\boldsymbol{\alpha} \models n$, is a formal composition $\alpha^{(1)} \oplus \cdots \oplus \alpha^{(k)}$, where $\alpha^{(i)} \models n_i$ for positive integers n_i's with $n_1 + \cdots + n_k = n$.
- Norton classified all projective indecomposable $H_n(0)$-modules $\mathbf{P}_\alpha$ and simple $H_n(0)$-modules $\mathbf{F}_\alpha := \mathbf{P}_\alpha/\mathrm{rad}(\mathbf{P}_\alpha)$ for every $\alpha \models n$.
- The *quasisymmetric characteristic* is defined by

$$\mathrm{ch} : \bigoplus_{n \ge 0} \mathcal{G}_0(H_n(0)) \to \mathrm{QSym}, \quad [\mathbf{F}_\alpha] \mapsto F_\alpha.$$

where F_α is the fundamental quasisymmetric function.
- We consider the following indecomposable $H_n(0)$-modules:
  - $\mathbf{S}^\sigma_{\alpha,E}$ (Tewari–van Willigenburg, 2015)
  - $\mathcal{V}_\alpha$ (Berg–Bergeron–Saliola–Serrano–Zabrocki, 2015)
  - X_α (Searles, 2020)

These modules come from certain bases of QSym:
  - $\sum_E [\mathbf{S}^{\mathrm{id}}_{\alpha,E}] \mapsto \mathcal{S}_\alpha$ (the quasisymmetric Schur function, 2011)
  - $[\mathcal{V}_\alpha] \mapsto \mathfrak{S}^*_\alpha$ (the dual immaculate function, 2014)
  - $[X_\alpha] \mapsto \mathcal{E}_\alpha$ (the extended Schur function, 2019).

The $H_n(0)$-module $\mathbf{S}^\sigma_\alpha$

- Given $\alpha \models n$ and $\sigma \in \Sigma_{\ell(\alpha)}$, a *standard permuted composition tableau* (SPCT) of shape α and type σ is $\tau : \mathtt{cd}(\alpha) \to [n]$ such that
  - the entries are all distinct,
  - the standardized 1st column word $\mathrm{st}(w^1(\tau))$ is σ,
  - the entries along the rows decrease from left to right,
  - (the *triple condition*) if $i < j$ and $\tau_{i,k} > \tau_{j,k+1}$, then $(i, k+1) \in \mathtt{cd}(\alpha)$ and $\tau_{i,k+1} > \tau_{j,k+1}$.
- Let $\mathrm{SPCT}^\sigma(\alpha)$ be the set of SPCTx of shape α and type σ. Let $\tau \in \mathrm{SPCT}^\sigma(\alpha)$ and $i < j$ in τ. We say that i and j are *attacking* in τ if either
  i and j are in the same column, or
  i and j are in adjacent columns, with j positioned upper-left of i.
- Given $\tau \in \mathrm{SPCT}^\sigma(\alpha)$, we say that $i \in \tau$ is a *descent* if $i+1$ lies weakly right of i in τ. A descent i is *attacking (resp., nonattacking)* if i and $i+1$ are (resp., not) attacking .
- The $H_n(0)$-action on the set of SPCTx of shape α and type σ is defined by

$$\pi_i \cdot \tau = \begin{cases} \tau & \text{if } i \text{ is not a descent,} \\ 0 & \text{if } i \text{ is an attacking descent,} \\ s_i \cdot \tau & \text{if } i \text{ is a nonattacking descent} \end{cases}$$

We denote by $\mathbf{S}^\sigma_\alpha$ the resulting $H_n(0)$-module.
- Let $\sim_\alpha$ on $\mathrm{SPCT}^\sigma(\alpha)$ be defined by

$$\tau_1 \sim_\alpha \tau_2 \quad \text{if and only if} \quad \mathrm{st}(w^i(\tau_1)) = \mathrm{st}(w^i(\tau_2)) \ \ \forall\, i$$

for $\tau_1, \tau_2 \in \mathrm{SPCT}^\sigma(\alpha)$. Let $\mathcal{E}^\sigma(\alpha)$ be the set of all equivalence classes under $\sim_\alpha$. Then

$$\mathbf{S}^\sigma_\alpha \cong \bigoplus_{E \in \mathcal{E}^\sigma(\alpha)} \mathbf{S}^\sigma_{\alpha,E}.$$

- Note that $\mathbf{S}^\sigma_{\alpha,E}$ is cyclically generated by a unique SPCT, called the *source tableau* (see Example I.).

The $H_n(0)$-module $\mathcal{V}_\alpha$

- Given $\alpha \models n$, a *standard immaculate tableau* (SIT) of shape α is $\mathcal{T} : \mathtt{cd}(\alpha) \to [n]$ such that
  - the entries are all distinct,
  - the entries in each row increase from left to right,
  - the entries in 1st column increase from top to bottom.
- The $H_n(0)$-action on $\mathrm{SIT}(\alpha)$ is defined by

$$\pi_i \cdot \mathcal{T} = \begin{cases} \mathcal{T} & \text{if } i \text{ is weakly below } i+1, \\ 0 & \text{if } i \text{ and } i+1 \text{ are in 1st column,} \\ s_i \cdot \mathcal{T} & \text{otherwise.} \end{cases}$$

We denote by $\mathcal{V}_\alpha$ the resulting $H_n(0)$-module.

The $H_n(0)$-module X_α

- Given $\alpha \models n$, a *standard extended tableau* (SET) of shape α is $\mathsf{T} : \mathtt{cd}(\alpha^{\mathrm{r}}) \to [n]$ such that
  - the entries are all distinct,
  - the entries in each row increase from left to right,
  - the entries in each column increase from bottom to top.
- The $H_n(0)$-action on $\mathrm{SET}(\alpha)$ is defined by

$$\pi_i \cdot \mathsf{T} = \begin{cases} \mathsf{T} & \text{if } i \text{ is strictly left of } i+1, \\ 0 & \text{if } i \text{ and } i+1 \text{ are in the same column,} \\ s_i \cdot \mathsf{T} & \text{if } i \text{ is strictly right of } i+1. \end{cases}$$

We denote by X_α the resulting $H_n(0)$-module.

Theorem I. Structures of $\mathbf{S}^\sigma_\alpha$

(a) For every class $E \in \mathcal{E}^\sigma(\alpha)$, $\mathbf{S}^\sigma_{\alpha,E}$ is indecomposable.
(b) If there is $1 \le i \le \ell(\alpha) - 1$ such that $\ell(\sigma s_i) < \ell(\sigma)$, then

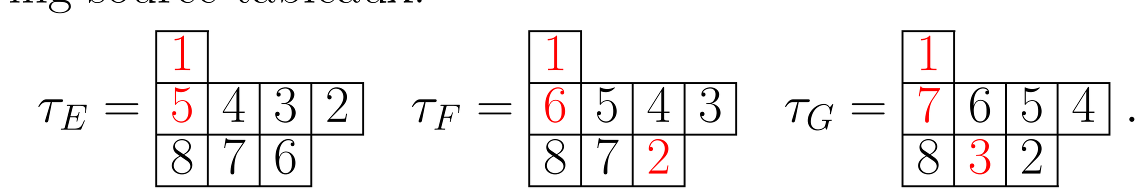
$$\mathrm{ch}([\mathbf{S}^\sigma_\alpha]) = \sum_{\alpha = \beta \bullet \pi_i} \mathrm{ch}([\mathbf{S}^{\sigma s_i}_\beta]).$$

And the quasi-Schur expansion of $\mathrm{ch}([\mathbf{S}^\sigma_\alpha])$ equals $\sum_{\alpha=\beta \bullet \pi_\sigma} \mathcal{S}_\beta$. Here $\bullet$ is the bubble sorting' action.
(c) For $E \in \mathcal{E}^\sigma(\alpha)$, let τ_E be the source tableau of E. Let $\mathrm{Des}(\tau_E) = \{d_1 < d_2 < \cdots < d_m\}$, $d_0 := 0$, and $d_{m+1} := n$. Then the generalized composition $\boldsymbol{\alpha}_E$ is defined as follows:
1. Let $\boldsymbol{\alpha}^{(1)} = (1^{d_1})$.
2. For $1 \le j \le m$, define $\boldsymbol{\alpha}^{(j+1)} = \begin{cases} \boldsymbol{\alpha}^{(j)} \odot (1^{d_{j+1}-d_j}) & \text{if } d_{j-1}+1 \text{ is weakly right of } d_j \text{ or } d_{j-1}+1 \text{ and } d_j \text{ are attacking,} \\ \boldsymbol{\alpha}^{(j)} \oplus (1^{d_{j+1}-d_j}) & \text{otherwise.} \end{cases}$
3. Set $\boldsymbol{\alpha}_E := \boldsymbol{\alpha}^{(m+1)}$.

We provide an essential $H_n(0)$-module epimorphism $\eta : \mathbf{P}_{\boldsymbol{\alpha}_E} \to \mathbf{S}^\sigma_{\alpha,E}$, thus $(\mathbf{P}_{\boldsymbol{\alpha}_E}, \eta)$ is a projective cover of $\mathbf{S}^\sigma_{\alpha,E}$.

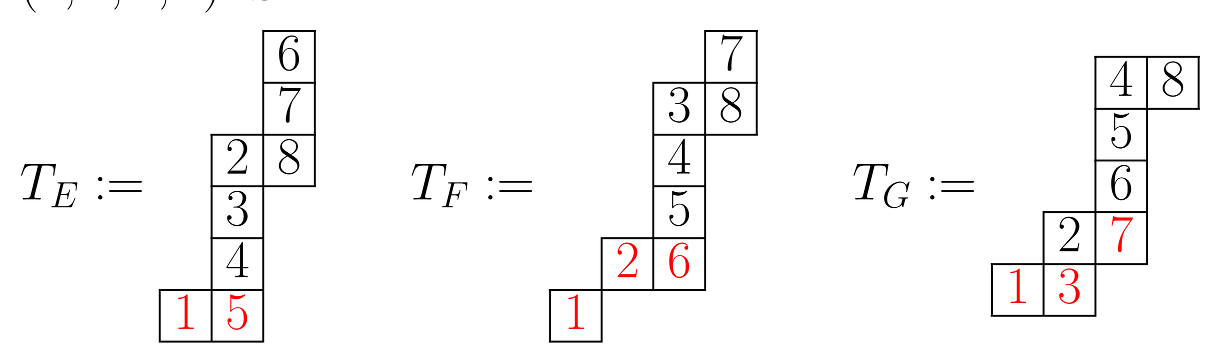
Example I. Structures of $\mathbf{S}^\sigma_\alpha$

(a) $\mathrm{ch}([\mathbf{S}^{132}_{(1,4,3)}]) = \mathrm{ch}([\mathbf{S}^{\mathrm{id}}_{(1,4,3)}]) + \mathrm{ch}([\mathbf{S}^{\mathrm{id}}_{(1,3,4)}])$
(b) Note that $|\mathcal{E}^{\mathrm{id}}((1,4,3))| = 3$. More precisely, let E, F and G be the equivalence classes of the following source tableaux:



canonical — noncanonical — noncanonical

Then $\boldsymbol{\alpha}_E = (1,4,3)$, $\boldsymbol{\alpha}_F = (1) \oplus (1,4,2)$, $\boldsymbol{\alpha}_G = (1,2,4,1)$. Set

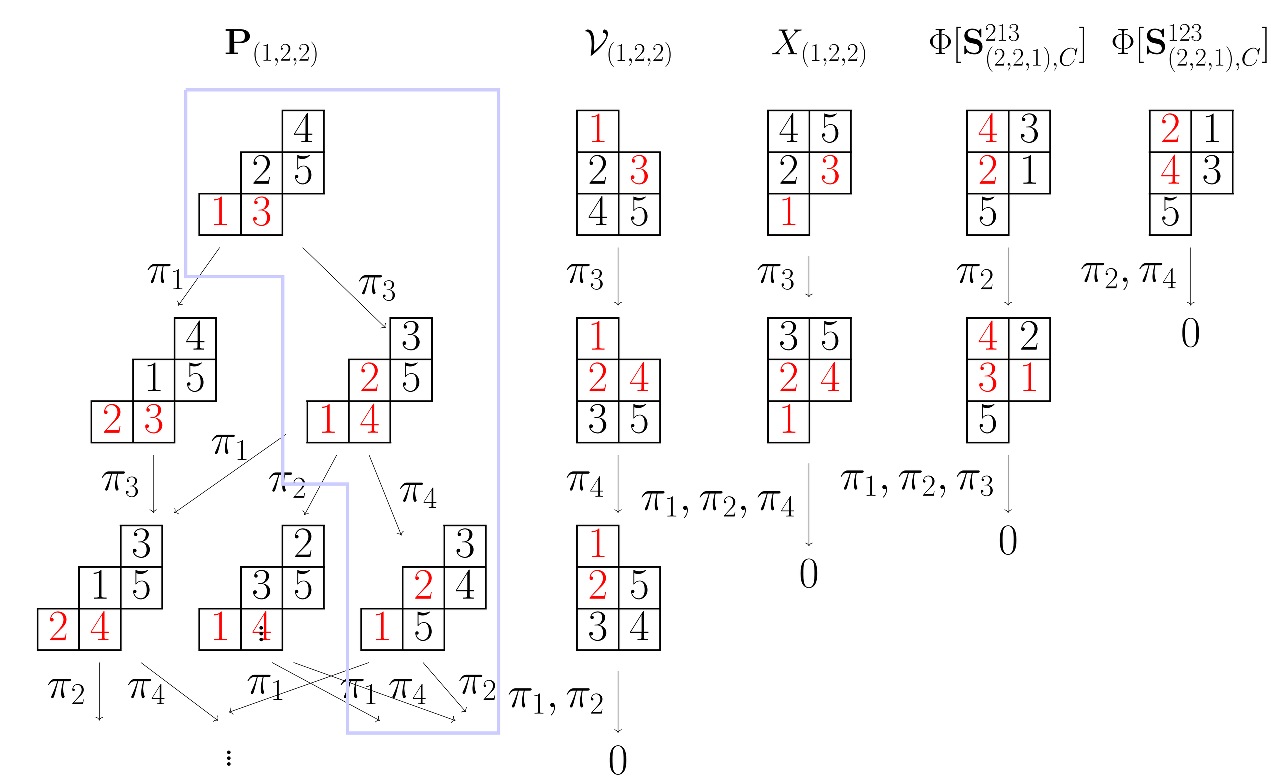


Then we have

$$\eta(T_E) = \tau_E, \ \eta(T_F) = \tau_F \ \text{ and } \ \eta(T_G) = \tau_G.$$

Example II. Relationships

$$\mathbf{P}_{(1,2,2)} \to \mathcal{V}_{(1,2,2)} \to X_{(1,2,2)} \to \Phi[\mathbf{S}^{213}_{(2,2,1),C}] \to \Phi[\mathbf{S}^{123}_{(2,2,1),C}]$$



Theorem II. Relationships between $H_n(0)$-modules arising from tableaux

We construct a series of surjections

$$\mathbf{P}_\alpha \to \mathcal{V}_\alpha \to X_\alpha \to \Phi[\mathbf{S}^\sigma_{\lambda(\alpha),C}] \to \Phi[\mathbf{S}^{\sigma s_{i_k}}_{\lambda(\alpha)\cdot s_{i_k},C}] \to \Phi[\mathbf{S}^{\sigma s_{i_k} s_{i_{k-1}}}_{\lambda(\alpha)\cdot s_{i_k} s_{i_{k-1}},C}] \to \cdots \to \Phi[\mathbf{S}^{\mathrm{id}}_{\lambda(\alpha)\cdot\sigma^{-1},C}].$$

Here,
- $\lambda(\alpha)$ is the partition obtained from α by arranging the parts of α,
- C is the canonical class in $\mathcal{E}^\sigma(\alpha)$,
- σ is any permutation in $\Sigma_{\ell(\alpha)}$ satisfying that $\lambda(\alpha) = \alpha^{\mathrm{r}} \cdot \sigma$,
- $\Phi[\mathbf{S}^\sigma_{\lambda(\alpha),C}]$ is the Φ-twist of $\mathbf{S}^\sigma_{\lambda(\alpha),C}$, where Φ is the automorphism of $H_n(0)$ given by $\Phi(\pi_i) = \pi_{n-i}$ for $i \in [n-1]$,
- $s_{i_1} \cdots s_{i_k}$ is a reduced expression of $\lambda(\sigma)$.