3-Stack-Sortable Permutations and Beyond

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Let S_n be the set of permutations of $[n]$.

West's stack-sorting map is the operator $s: S_n \to S_n$ defined recursively by

$$s(LnR) = s(L)\,s(R)\,n.$$

Example: $s(3521764) = s(3521)\,s(64)\,7 = s(3)\,s(21)\,5\,s(4)\,6\,7 = 3\,s(1)\,2\,5\,4\,6\,7 = 3125467.$

Definition: Say a permutation $\pi \in S_n$ is **t-stack-sortable** if $s^t(\pi) = 123\cdots n$. Write $\mathcal{W}_t(n) = \{t\text{-stack-sortable permutations in } S_n\}$ and $W_t(n) = |\mathcal{W}_t(n)|$.

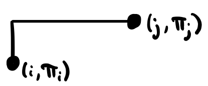
Theorem (Knuth, 1968): $\mathcal{W}_1(n) = \{231\text{-avoiding permutations in } S_n\}$ and $W_1(n) = C_n = \frac{1}{n+1}\binom{2n}{n}$.

Theorem (Zeilberger, 1992): $W_2(n) = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}$. ← Reproven and refined by several authors.

Úlfarsson described $\mathcal{W}_3(n)$ in terms of "decorated patterns," but the description does not seem helpful for enumeration. For a while, the values of $W_3(n)$ were only known for $n \le 13$.

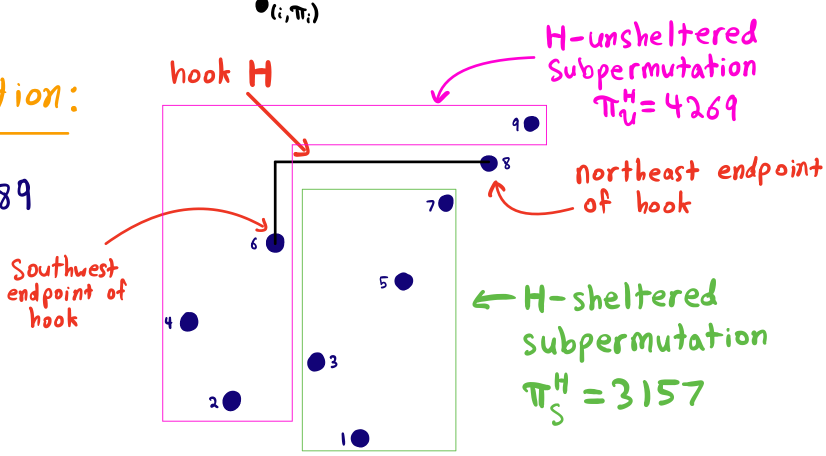
The key to understanding $W_3(n)$ is to study $|s^{-1}(\pi)|$, called the **fertility** of π.

The **plot** of π is the diagram showing the points (i, π_i) for all $i \in [n]$.

A **hook** of π is a shape  drawn in the plot for $i<j$ and $\pi_i < \pi_j$.

Example/Definition:

$\pi = 426315789$



Let $SW_i(\pi)$ be the set of hooks of π with southwest endpoint (i, π_i).

A descent of π is an index $d \in [n-1]$ such that $\pi_d > \pi_{d+1}$.

Let d^* be the largest descent of π.

Say a descent d is right-bound if $\pi_i < \pi_d$ for all $i \in \{d+1, \dots, d^*\}$.

Decomposition Lemma (D., 2020): If d is a right-bound descent of π, then

$$|s^{-1}(\pi)| = \sum_{H \in SW_d(\pi)} |s^{-1}(\pi_U^H)| \cdot |s^{-1}(\pi_S^H)|.$$

Applications:

- Generalizes to a broader setting and leads to a connection between stack-sorting and free probability theory.
- Can be used to prove that $|s^{-1}(s(\sigma))| \geq |s^{-1}(\sigma)|$ for all $\sigma \in S_n$.
- Leads to a new proof of Zeilberger's formula. Can be used to count 2-stack-sortable permutations according to descents and peaks.
- Leads to the following (ridiculous) recurrence for computing $W_3(n)$:

Theorem (D., 2020): Define numbers $B_{\geq \ell}^{(g)}(n)$ by:

$$B_{\geq \ell}^{(0)}(n) = 0, \quad B_{\geq \ell}^{(g)}(1) = \begin{cases} 0 & \text{if } g \neq 2 \\ C_{\ell+1} & \text{if } g = 2, \end{cases} \quad \text{and}$$

Catalan

$$B_{\geq \ell}^{(g)}(n+1) = \sum_{j=1}^{\ell} \left(\sum_{a=2}^{n} \sum_{b=\max\{2, g-a\}}^{g-1} \sum_{i=a-1}^{n-b+1} B_{\geq j-1}^{(a)}(i) B_{\geq \ell-j+1}^{(b)}(n-i) + B_{\geq j-1}^{(g-1)}(n)\, C_{\ell-j+1} \right) + B_{\geq \ell+1}^{(g-1)}(n)$$

for $n, g \geq 1$ and $\ell \geq 0$. Then

$$W_3(n) = \sum_{g=1}^{n+1} B_{\geq 0}^{(g)}(n).$$

Elvey Price, Guttmann, and I have now computed $W_3(n)$ for $n \leq 1000$.

We found that $\lim_{n\to\infty} W_3(n)^{1/n} \geq 9.4854$, disproving Bóna's conjecture that $W_3(n) \leq \binom{4n}{n}$.

We conjecture that $\lim_{n\to\infty} W_3(n)^{1/n} \approx 9.69963634535$.

The best known upper bound for this limit is $\lim_{n\to\infty} W_3(n)^{1/n} \leq 12.53$.

Open Problem: Improve this upper bound.