Minkowski Summands of Products of Simplicies

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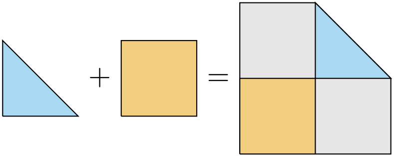
Minkowski sum

The *Minkowski sum* of two sets in a common vector space is the union of the sums of their elements.

$$A + B := \{a + b : a \in A, b \in B\}$$



What are possible summands of some set?

Weak Minkowski summands

We instead want weak Minkowski summands, because after modding out by translation, these form a cone.

$$\{Q : \exists R, \lambda \, s.t. Q + R = \lambda P\}$$

This cone is called the *type cone* or *nef cone*. Modding out by scalars gives the *type polytope*.

https://www.geogebra.org/m/szz3hhug

Type cone parameterizations

There are three ways to parameterize the type cone. The first two are due to Shephard.

- Facet height

The parameters are heights for every facet. Constraints come from preserving combinatorial type.

- Edge length

The parameters are lengths for every edge. Constraints come from balancing conditions around each cycle.

- McMullen's representation

Our main focus, and hard to summarize.

McMullen's Theorem

Theorem (McMullen)

Let P be a polytope, $\mathcal{A} = \{a_1, \ldots, a_m\}$ be the vertex set of P° and $Gale(\mathcal{A}) = \{b_1, \ldots, b_m\}$ be a Gale transform for $\mathcal{A}$. Then the type polytope of P is combinatorially equivalent to

$$\bigcap_S Conv(b_i : b_i \in S)$$

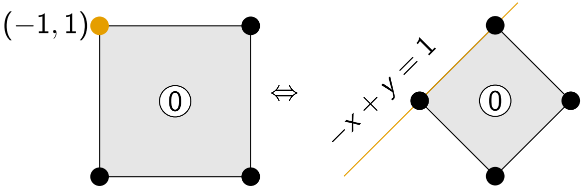
where the intersection is over all cofacets S of $\mathcal{A}$.

Polar dual

The *polar dual* of a set is defined by an inversion.

$$P^\circ := \{x : x \cdot y < 1 \, \forall y \in P\}$$

For a polytope, the polar dual exchanges the vertices and facets.



Gale dual

Given m points in a d dimensional space, the Gale dual is m points in $m - d - 1$ dimensional space.

Proposition

Set P a polytope, and $G : V(P) \to R^{v-d-1}$ a Gale dual of the vertices of P. For some collection $W \subset V(P)$, there is a face σ with exactly vertices W if and only if $G(V(P) \setminus W)$ contains the origin in its convex hull.

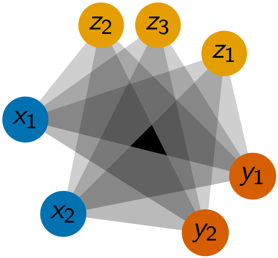
This proposition motivates the defintion of a *coface* of σ, the set of vertices which are not contained in σ.

Rainbows and Simplices

Let P be a product of simplices $\Delta^{a_1} \star \Delta^{a_2} \star \ldots \star \Delta^{a_k}$.

The vertices of P° are in bijection with the vertices of the Δ^{a_i}.
The cofacets of P° contain one vertex from each Δ^{a_i}.

In the Gale dual, each cofacet contains the origin.



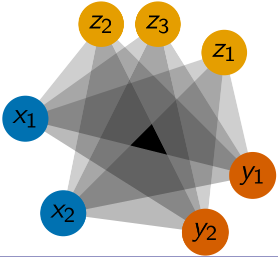
Intersection?

Theorem (McMullen)

The type polytope of P is combinatorially equivalent to

$$\bigcap_S Conv(b_i : b_i \in S)$$

where the intersection is over all cofacets S of $\mathcal{A}$.

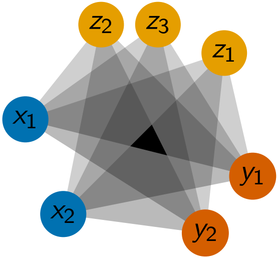


Main result

Theorem (Castillo, D., Goeckner, Ross, Ying)

The intersection of the cofacets is a simplex.

Proof: The intersection is a polytope, each facet is rainbow colored, there's a facet for each rainbow coloring, there can't be more than one facet of each rainbow color.



Consequences

Theorem (Castillo, D., Goeckner, Ross, Ying)

The type cone of a product of simplices is a simplex.

Corollary

The space of Minkowski summands of a d-cube is d dimensional.

The other parameterizations give $2d$ or $d2^d$ dimensional parameterizations.

Besides polytopes where the type cone is linear, we know of no other polytope where the type cone combinatorics does not depend on the polytope geometry.