Box-ball systems and RSK tableaux

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Ben Drucker (presenter)

Eli Garcia

Rose Silver

Aubrey Rumbolt

Emily Gunawan (faculty mentor)

Box-Ball System and Example

A box-ball system is a collection of discrete time states each containing elements of a permutation, π in one-line notation. We can move forward in this system by rearranging π using the following rule: Place π in a strip of infinite boxes. (This corresponds to the $t = 0$ state of the system.) Then, move each element of π to the nearest empty box to its right:

$t = 0$	4	5	2	3	6	1							...
	4	5	2	3	6		1						...
	4	5		3	6	2	1						...
	4	5			6	2	1	3					...
		5	4		6	2	1	3					...
			4	5	6	2	1	3					...
$t = 1$			4	5		2	1	3	6				...

Box-Ball System and Example

To continue the analysis, keep making BBS moves until we reach a "steady state" (i.e., the system decomposes into *solitons*)

$t = 0$	4	5	2	3	6	1														⋯
$t = 1$			4	5		2	1	3	6											⋯
$t = 2$					4	5	2			1	3	6								⋯
$t = 3$							4	2	5				1	3	6					⋯
$t = 4$								4		2	5					1	3	6		⋯

Then, create a (not necessarily standard) tableau by stacking solitons:

Soliton decomposition $\begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$ with shape $(3, 2, 1)$.

RSK algorithm

The Robinson–Schensted–Knuth (RSK) insertion algorithm is a bijection

$$\pi \mapsto (P(\pi), Q(\pi))$$

from S_n onto pairs of size-n standard tableaux of equal shape.

Example

Let $\pi = \mathbf{452361}$. $P(\pi) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$ and $Q(\pi) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$.

Selected Results Involving Insertion Tableaux

Theorem

The following are equivalent:

1. $\mathrm{SD}(\pi) = \mathrm{P}(\pi)$.
2. $\mathrm{SD}(\pi)$ is a standard tableau.
3. $\operatorname{sh}\mathrm{SD}(\pi) = \operatorname{sh}\mathrm{P}(\pi)$.

$\star(3 \implies 2$ *due to Darij Grinberg*$)$

Selected Results Involving Steady-State Times

Reading Words

The *reading word* of a tableau is the permutation formed by concatenating the rows of the tableau from bottom to top.
Example: The reading word of $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \multicolumn{1}{|c|}{5} \\ \cline{1-1}\end{array}$ is 53412.

Theorem: Classification of permutations with steady-state time of $t = 0$

A permutation r has a box-ball steady-state time of $t = 0$ if and only if r is the reading word of a standard tableau.

Theorem: Some permutations with steady-state time of $t = 1$

Let r be the reading word of a standard tableau P. If r' is the permutation resulting from the swapping of a special pattern of elements in r, then r' has a box-ball steady-state time of $t = 1$. (r' is the result of performing a *Knuth move* on r.)

Selected Results Involving Steady-State Times

Theorem: Permutations with special recording tableau

If $n \geq 5$, every permutation with recording tableau equal to

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & \cdots & n-2 & n-1 \\ \hline 3 & 4 \\ \cline{1-2} n \\ \cline{1-1}\end{array}$$

has steady-state time $n - 3$.

Conjecture: $n - 3$

The steady-state time for a permutation in S_n is no more than $n - 3$, for $n \geq 4$.

Thank You!
Questions?