qRSt: A probabilistic Robinson–Schensted correspondence for Macdonald polynomials

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Schur polynomials and insertion algorithms

A *semistandard Young tableau (SSYT)* of shape λ is a filling of the cells of the Young diagram of λ with positive integers, such that

- rows weakly increase from left to right
- columns strictly increase from bottom to top.

For example, here is a SSYT of shape $(6, 4, 1)$:

7					
2	2	5	8		
1	1	1	3	3	5

A *standard Young tableau (SYT)* of shape λ is an SSYT that uses each of the numbers $1, 2, \ldots, n = |\lambda|$ exactly once. Let f_λ denote the number of standard Young tableaux of shape λ.

Let $X = (x_1, x_2, \ldots)$. The *Schur function* $s_\lambda(X)$ is defined by

$$s_\lambda(X) = \sum_{T \in \mathrm{SSYT}(\lambda)} X^T \qquad \left(X^T = x_1^{\#1's \text{ in } T} x_2^{\#2's \text{ in } T} \cdots \right).$$

Theorem (Cauchy identity)

The Schur functions in two sets of variables $X = (x_1, x_2, \ldots)$ and $Y = (y_1, y_2, \ldots)$ satisfy the identity

$$\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \sum_\lambda s_\lambda(X) s_\lambda(Y).$$

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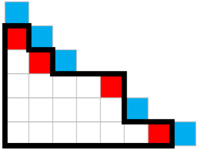
Taking coefficients of $x_1 \cdots x_n y_1 \cdots y_n$ gives the classical identity

$$n! = \sum_{|\lambda| = n} (f_\lambda)^2.$$

We will now consider a large class of bijections between permutations and pairs of standard Young tableaux of the same shape. These algorithms appeared in a 1986 paper by Fomin (and in an unpublished paper of Garsia–McLarnan from 1987).

For a partition λ, define

$$\mathcal{U}(\lambda) = \{\lambda + \square\}, \qquad \mathcal{D}(\lambda) = \{\lambda - \square\}.$$



The up and down operators on the vector space spanned by partitions are defined by

$$U\lambda = \sum_{\nu \in \mathcal{U}(\lambda)} \nu, \qquad D\lambda = \sum_{\mu \in \mathcal{D}(\lambda)} \mu.$$

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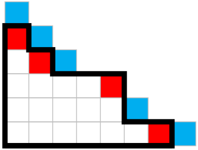

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These operators satisfy

$$DU - UD = I.$$

The identity $n! = \sum_{|\lambda|=n} (f_\lambda)^2$ is easily derived from this relation:

$$\sum_{|\lambda|=n} (f_\lambda)^2 \, \emptyset = D^n U^n \emptyset = n! \, \emptyset.$$

definition of SYT

n applications of commutation relation

Fomin's insight: a bijective proof of the relation $DU - UD = I$ yields a bijective proof of the identity $n! = \sum_{|\lambda|=n}(f_\lambda)^2$.

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The relation $DU - UD = I$ says two things:

$$\begin{array}{ll} |\mathcal{U}(\lambda) \cap \mathcal{U}(\rho)| = |\mathcal{D}(\lambda) \cap \mathcal{D}(\rho)| & \text{for } \lambda \neq \rho, \\ |\mathcal{U}(\lambda)| = |\mathcal{D}(\lambda)| + 1 & \text{for all } \lambda. \end{array}$$

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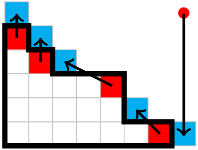
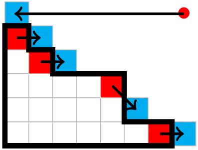
For the second family of identities, there are many possible bijective proofs. Define a set of *local growth rules* to be a family of bijections

$$\mathcal{F} = \{F_\lambda\}, \qquad F_\lambda : \{\bullet\} \cup \mathcal{D}(\lambda) \to \mathcal{U}(\lambda).$$

There are two natural sets of local growth rules, $\mathcal{F}^{\mathrm{row}}$ and $\mathcal{F}^{\mathrm{col}}$:



F_λ^{row}



F_λ^{col}

Let T be a SSYT with no repeated entries, and k a number not in T. Let $T^{<j}$ be the shape formed by the entries $< j$ in T. Given $\mathcal{F} = \{F_\lambda\}$, define the *$\mathcal{F}$-insertion* of k into T as follows:

- **Initial insertion step** Place k in the cell $F_{T^{<k}}(\bullet)/T^{<k}$, possibly bumping a larger number from this cell.
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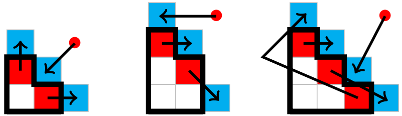
Let's compute

5	9	
3	6	8
1	2	7

$\xleftarrow{\mathcal{F}} 4$,

where $\mathcal{F}$ contains these three bijections:



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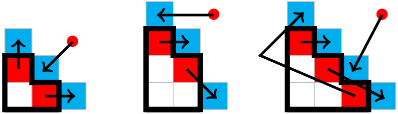

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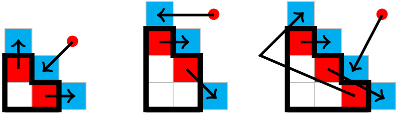

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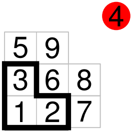

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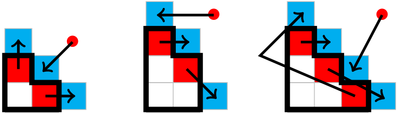

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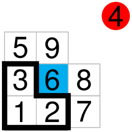

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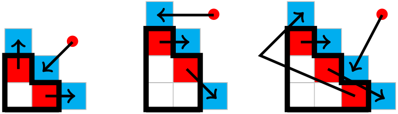

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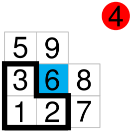

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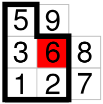

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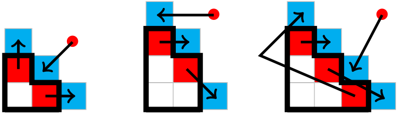

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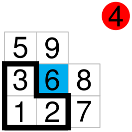

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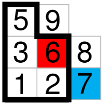

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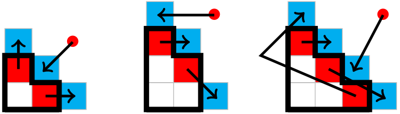

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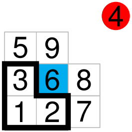

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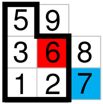

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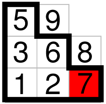

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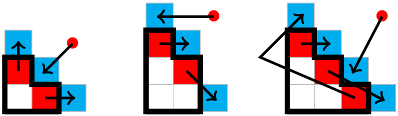

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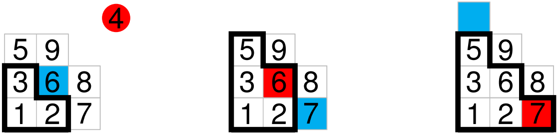

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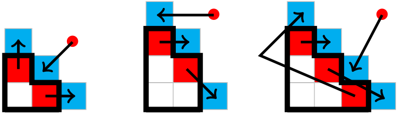

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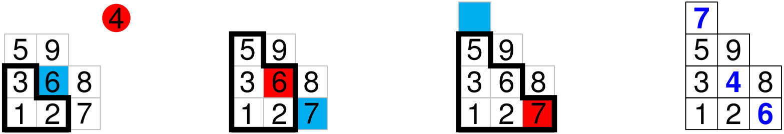

Insert 4 **Bump 6** **Bump 7** End result

$\mathcal{F}$-insertion gives rise to a bijection $\mathrm{RS}_{\mathcal{F}}\colon w_1 \cdots w_n \mapsto (P, Q)$ from permutations to pairs of SYTs of the same shape, as follows:

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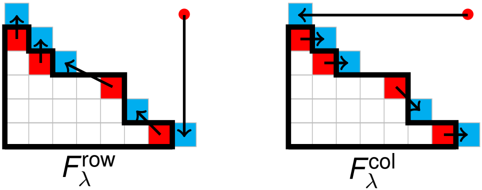
In particular, the families $\mathcal{F}^{\mathrm{row}}$ and $\mathcal{F}^{\mathrm{col}}$ give rise to the row insertion and column insertion versions of Robinson–Schensted.



Theorem (Fomin '86)

Each bijection $\mathrm{RS}_{\mathcal{F}}$ *satisfies the symmetry property*

$$\mathrm{RS}_{\mathcal{F}}(w) = (P, Q) \iff \mathrm{RS}_{\mathcal{F}}(w^{-1}) = (Q, P).$$

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RS and RSK have applications to:

- combinatorics (e.g., enumeration of plane partitions)
- representation theory (e.g., crystal bases)
- algebraic geometry (e.g., Springer fibers in the flag variety)
- probability theory (e.g., length of the longest increasing subsequence of a random permutation)

Macdonald polynomials and probabilistic insertion algorithms

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  - Jack symmetric functions ($t = q^\alpha, q \to 1$)
- There are explicit combinatorial formulas for the $P_\lambda(X; q, t)$ and for the dual basis $Q_\lambda(X; q, t)$:

$$P_\lambda(X; q, t) = \sum_{T \in \mathrm{SSYT}(\lambda)} \psi_T(q, t) X^T,$$

$$Q_\lambda(X; q, t) = \sum_{T \in \mathrm{SSYT}(\lambda)} \varphi_T(q, t) X^T,$$

where ψ_T, φ_T are rational functions in q, t.

Theorem (Generalized Cauchy identity)

$$\prod_{i,j\geq 1}\prod_{k\geq 0}\frac{(1-tx_iy_jq^k)}{(1-x_iy_jq^k)}=\sum_{\lambda}P_\lambda(X;q,t)Q_\lambda(Y;q,t).$$

Macdonald proved this result using his original definition of the P_λ and Q_λ in terms of an inner product.

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As in the Schur case, it is natural to start by considering the coefficients of $x_1\cdots x_n y_1\cdots y_n$ on either side:

$$\frac{(1-t)^n}{(1-q)^n}n!=\sum_{|\lambda|=n}\sum_{(P,Q)\in \mathrm{SYT}(\lambda)\times\mathrm{SYT}(\lambda)}\psi_P(q,t)\varphi_Q(q,t).$$

$$n! = \sum_{|\lambda|=n} \sum_{(P,Q)\in \mathrm{SYT}(\lambda)\times \mathrm{SYT}(\lambda)} \frac{(1-q)^n}{(1-t)^n} \psi_P(q,t)\varphi_Q(q,t).$$

$$n! = \sum_{|\lambda|=n} \sum_{(P,Q)\in \mathrm{SYT}(\lambda)\times \mathrm{SYT}(\lambda)} \frac{(1-q)^n}{(1-t)^n} \psi_P(q,t)\varphi_Q(q,t).$$

Example ($n = 2$)

	$\sigma \in S_2$	P, Q	$\frac{(1-q)^2}{(1-t)^2}\psi_P(q,t)\varphi_Q(q,t)$
1	12	$\boxed{1}\boxed{2}\,,\ \boxed{1}\boxed{2}$	$\frac{(1-t)(1-q^2)}{(1-q)(1-qt)}$
1	21	$\begin{array}{\|c\|}\hline 2 \\ \hline 1 \\ \hline \end{array}\,,\ \begin{array}{\|c\|}\hline 2 \\ \hline 1 \\ \hline \end{array}$	$\frac{(1-q)(1-t^2)}{(1-t)(1-qt)}$

So the above identity says that

$$1 + 1 = \frac{(1-t)(1-q^2)}{(1-q)(1-qt)} + \frac{(1-q)(1-t^2)}{(1-t)(1-qt)}.$$

This cannot be proved by a bijection from permutations to SYTs!

Definition (taken from Bufetov–Matveev '18, Bufetov–Petrov '19)

Let X, Y be finite sets equipped with weight functions $\omega\colon X \to k, \overline{\omega}\colon Y \to k$, k a field. A *probabilistic bijection* from (X, ω) to $(Y, \overline{\omega})$ is a pair of functions $\mathcal{P}, \overline{\mathcal{P}}\colon X \times Y \to k$ satisfying

- $\sum_{y \in Y} \mathcal{P}(x, y) = 1$ for $x \in X$
- $\sum_{x \in X} \overline{\mathcal{P}}(x, y) = 1$ for $y \in X$
- $\omega(x)\mathcal{P}(x, y) = \overline{\mathcal{P}}(x, y)\overline{\omega}(y)$ for $x \in X, y \in Y$.

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- $\sum_{y \in Y} \mathcal{P}(x \to y) = 1$ for $x \in X$
- $\sum_{x \in X} \overline{\mathcal{P}}(x \leftarrow y) = 1$ for $y \in X$
- $\omega(x)\mathcal{P}(x \to y) = \overline{\mathcal{P}}(x \leftarrow y)\overline{\omega}(y)$ for $x \in X, y \in Y$.

The existence of a probabilistic bijection from (X, ω) to $(Y, \overline{\omega})$ implies the identity

$$\sum_{x \in X} \omega(x) = \sum_{y \in Y} \overline{\omega}(y).$$

This notion

- is equivalent to a joint distribution on $X \times Y$ with marginals proportional to ω and $\overline{\omega}$ (assuming the equality of weighted sums)
- appears in the definition of the Wasserstein (or earth mover's) distance between two probability distributions.

There are several probabilistic one-parameter deformations of RS and RSK in the literature:

q-RS (row)	Borodin–Petrov '16
q-RS (column)	O'Connell–Pei '13, Pei '14
t-RS (column)	Bufetov–Petrov '15
q-RSK (row)	Matveev–Petrov '17
q-RSK (column)	Matveev–Petrov '17
t-RSK (column)	Bufetov–Matveev '18

These algorithms have been used to analyze interacting particle systems such as the ASEP, q-TASEP, and q-PushTASEP, as well as the stochastic six-vertex model and the distribution of matrices over a finite field.

Our contribution is the construction of a (q, t)-RS algorithm, which we call $q\mathrm{RS}t$.

We saw that a family of bijections

$$F_\lambda \colon \{\bullet\} \cup \mathcal{D}(\lambda) \to \mathcal{U}(\lambda)$$

produces a bijection between permutations and pairs of standard Young tableaux of the same shape.

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In exactly the same way, a probabilistic bijection proving the identity

$$n! = \sum_{|\lambda|=n} \sum_{(P,Q)\in \mathrm{SYT}(\lambda)\times \mathrm{SYT}(\lambda)} \frac{(1-q)^n}{(1-t)^n} \psi_P(q,t)\varphi_Q(q,t)$$

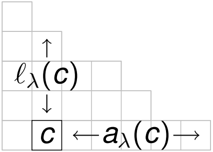
can be obtained from a family of probabilistic bijections

$$\mathcal{P}_\lambda \colon (\{\bullet\} \cup \mathcal{D}(\lambda), \omega_\lambda) \to (\mathcal{U}(\lambda), \overline{\omega}_\lambda),$$

where $\omega_\lambda(\bullet) = 1$, and $\omega_\lambda(\mu), \overline{\omega}_\lambda(\nu)$ are rational functions in q, t related to the tableau formulas for the Macdonald polynomials.

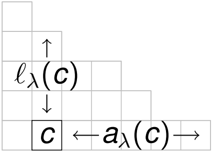
The probabilities $\mathcal{P}_\lambda(\bullet \to \nu)$ govern the initial insertion step, and the probabilities $\mathcal{P}_\lambda(\mu \to \nu)$ govern the bumping steps.

Arms, legs, hook lengths, and probabilities



The cell $c \in \lambda$ has

- *arm-length* $a_\lambda(c) = 5$
- *leg-length* $\ell_\lambda(c) = 3$
- *hook-length* $h_\lambda(c) = a_\lambda(c) + \ell_\lambda(c) + 1 = 9.$



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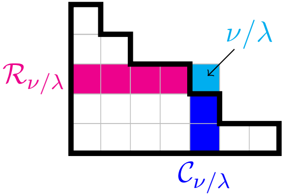
For $\nu \in \mathcal{U}(\lambda)$, define

$$\alpha_{\nu/\lambda} = \prod_{c \in \mathcal{R}_{\nu/\lambda}} \frac{[h_\lambda(c)]^\ell}{[h_\nu(c)]^\ell} \prod_{c \in \mathcal{C}_{\nu/\lambda}} \frac{[h_\lambda(c)]^a}{[h_\nu(c)]^a},$$



where $$[h_\lambda(c)]^\ell = 1 - q^{a_\lambda(c)} t^{\ell_\lambda(c)+1},$$
$$[h_\lambda(c)]^a = 1 - q^{a_\lambda(c)+1} t^{\ell_\lambda(c)}.$$

It is easy to see that

$$\lim_{q=t\to 1} \alpha_{\nu/\lambda} = \frac{H_\lambda}{H_\nu}, \qquad \text{where } H_\lambda = \prod_{c \in \lambda} h_\lambda(c).$$

Theorem (Aigner–F.)

The following probabilities define a probabilistic bijection from $(\{\bullet\} \cup \mathcal{D}(\lambda), \omega_\lambda)$ to $(\mathcal{U}(\lambda), \overline{\omega}_\lambda)$:

$$\mathcal{P}_\lambda(\bullet \to \nu) = t^{r_\nu - 1} \alpha_{\nu/\lambda},$$

$$\mathcal{P}_\lambda(\mu \to \nu) = t^{r_\nu - r_\mu - 1} \frac{\alpha_{\nu/\lambda}}{\alpha_{\lambda/\mu}} \frac{(1-q)(1-t)}{(1 - q^{c_\mu - c_\nu} t^{r_\nu - r_\mu})(1 - q^{c_\mu - c_\nu + 1} t^{r_\nu - r_\mu - 1})},$$

where r_ν, c_ν (resp., r_μ, c_μ) are the indices of the row and column containing the cell ν/λ (resp., λ/μ).

We prove this theorem by defining backward probabilities $\overline{\mathcal{P}}_\lambda(\bullet \leftarrow \nu)$ and $\overline{\mathcal{P}}_\lambda(\mu \leftarrow \nu)$ in a similar way, so that the relation

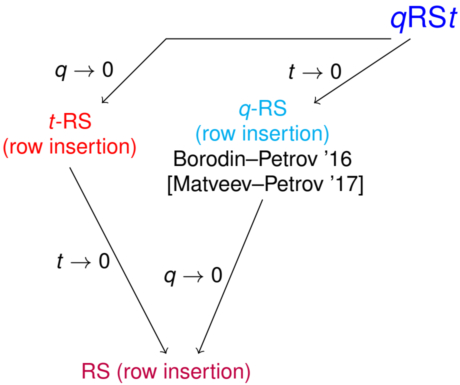
$$\omega_\lambda(\kappa) \mathcal{P}_\lambda(\kappa \to \nu) = \overline{\mathcal{P}}_\lambda(\kappa \leftarrow \nu) \overline{\omega}_\lambda(\nu)$$

is immediate. We then use Lagrange interpolation to prove that the $\mathcal{P}_\lambda$ and $\overline{\mathcal{P}}_\lambda$ sum to 1.

Degenerations of $q\mathrm{RS}t$

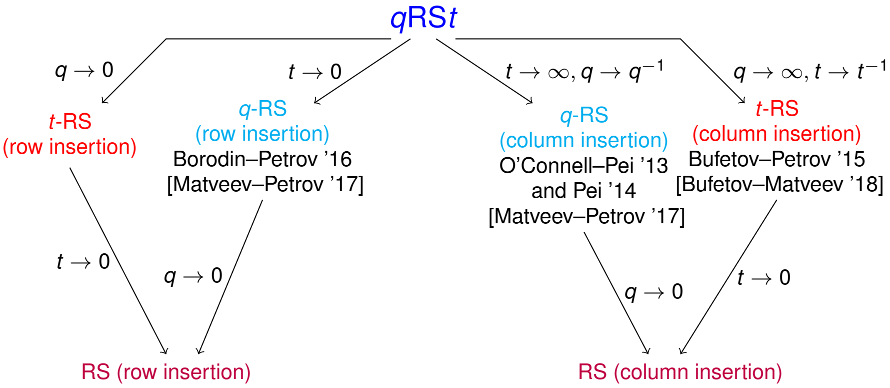
qRSt

Degenerations of $q\mathrm{RS}t$



Macdonald polynomials
Hall–Littlewood polynomials
q-Whittaker polynomials
Schur polynomials

Degenerations of $q\mathrm{RS}t$



Macdonald polynomials
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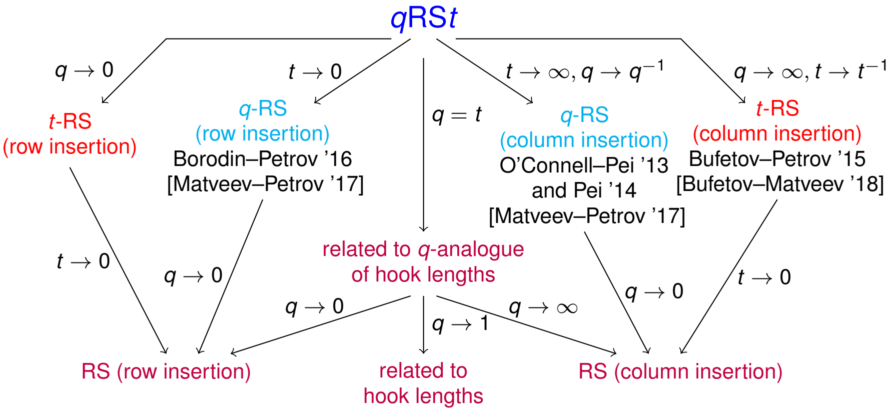
Degenerations of qRSt

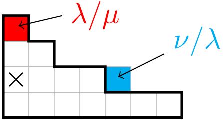


Macdonald polynomials
Hall–Littlewood polynomials
q-Whittaker polynomials
Schur polynomials

In the limit $q = t \to 1$, the probabilities degenerate to

$$\mathcal{P}_\lambda(\bullet \to \nu) = \frac{H_\lambda}{H_\nu}, \qquad \mathcal{P}_\lambda(\mu \to \nu) = \frac{(H_\lambda)^2}{H_\mu H_\nu} \frac{1}{(h_\lambda(c_{\mu,\nu}))^2},$$

where $c_{\mu,\nu}$ is the cell marked $\times$:



There are numerous proofs that the ratios H_λ / H_ν are a probability distribution on $\mathcal{U}(\lambda)$. For example, these probabilities arise from a version of the Greene–Nijenhuis–Wilf hook walk. (In fact, the $\mathcal{P}(\bullet \to \nu)$ arise from a (q, t)-hook walk similar to that of Garsia–Haiman '98.)

Problem

For fixed $\mu \in \mathcal{D}(\lambda)$, the expressions

$$\frac{(H_\lambda)^2}{H_\mu H_\nu} \frac{1}{(h_\lambda(c_{\mu,\nu}))^2} = \frac{|\lambda|}{|\lambda| + 1} \frac{f_\mu f_\nu}{(f_\lambda)^2} \frac{1}{(h_\lambda(c_{\mu,\nu}))^2}$$

are a probability distribution on $\mathcal{U}(\lambda)$. Find a combinatorial or probabilistic explanation of this fact.

Problem

Extend $q\mathrm{RS}t$ to a (q,t)-RSK algorithm which proves the full generalized Cauchy identity. Ideally, this map should

- *degenerate to RSK at $q = t = 0$*
- *degenerate to the Burge correspondence at $q = t \to \infty$*
- *degenerate to the known q-RSK and t-RSK maps mentioned above*
- *be given by actual probabilities when $q, t \in [0,1)$ or $q, t \in (1, \infty)$.*

We know how to extend $q\mathrm{RS}t$ from permutations to words, but the general biword/non-negative integer matrix setting seems to be much more challenging.

Thanks for listening!

Proof that $\mathcal{P}_\lambda, \overline{\mathcal{P}}_\lambda$ are probabilities (sketch)

Suppose $|\mathcal{U}(\lambda)| = d + 1$. Label the elements of $\mathcal{U}(\lambda)$ and $\mathcal{D}(\lambda)$

$$\lambda^{(+0)}, \ldots, \lambda^{(+d)}, \qquad \lambda^{(-1)}, \ldots, \lambda^{(-d)},$$

and set

$$p_{0,s} = \mathcal{P}_\lambda(\lambda \to \lambda^{(+s)}), \qquad p_{r,s} = \mathcal{P}_\lambda(\lambda^{(-r)} \to \lambda^{(+s)}).$$

There are formulas

$$p_{0,s} = \frac{\prod_{i=1}^{d}(q^{x_s} t^{y_s} - q^{x_{i-1}} t^{y_i})}{\prod_{\substack{i=0 \\ i \neq s}}^{d}(q^{x_s} t^{y_s} - q^{x_i} t^{y_i})}, \quad p_{r,s} = \prod_{\substack{i=0 \\ i \neq s}}^{d} \frac{q^{x_{r-1}+1} t^{y_r - 1} - q^{x_i} t^{y_i}}{q^{x_s} t^{y_s} - q^{x_i} t^{y_i}} \prod_{\substack{i=1 \\ i \neq r}}^{d} \frac{q^{x_s} t^{y_s} - q^{x_{i-1}} t^{y_i}}{q^{x_{r-1}+1} t^{y_r - 1} - q^{x_{i-1}} t^{y_i}}$$

where x_i, y_i are non-negative integers determined by λ. (There are very similar formulas for the backward probabilities $\overline{p}_{0,s}, \overline{p}_{r,s}$.)

There are formulas

$$p_{0,s} = \frac{\prod_{i=1}^{d}(q^{x_s}t^{y_s} - q^{x_{i-1}}t^{y_i})}{\prod_{\substack{i=0\\ i\neq s}}^{d}(q^{x_s}t^{y_s} - q^{x_i}t^{y_i})}, \quad p_{r,s} = \prod_{\substack{i=0\\ i\neq s}}^{d} \frac{q^{x_{r-1}+1}t^{y_r-1} - q^{x_i}t^{y_i}}{q^{x_s}t^{y_s} - q^{x_i}t^{y_i}} \prod_{\substack{i=1\\ i\neq r}}^{d} \frac{q^{x_s}t^{y_s} - q^{x_{i-1}}t^{y_i}}{q^{x_{r-1}+1}t^{y_r-1} - q^{x_{i-1}}t^{y_i}}.$$

To prove, for example, that $\sum_{s=0}^{d} p_{r,s} = 1$, we write $p_{r,s}$ in the form

$$p_{r,s} = \prod_{\substack{i=0\\ i\neq s}}^{d} \frac{b_r - a_i}{a_s - a_i} \prod_{\substack{i=1\\ i\neq r}}^{d} \frac{a_s - b_i}{b_r - b_i}.$$

Let $f(x) = \prod_{\substack{i=1\\ i\neq r}}^{d}(x - b_i)$. By Lagrange interpolation at the points $a_0, \ldots, a_d$,

$$\prod_{\substack{i=1\\ i\neq r}}^{d}(x - b_i) = \sum_{s=0}^{d} \prod_{\substack{i=1\\ i\neq r}}^{d}(a_s - b_i) \prod_{\substack{i=0\\ i\neq s}}^{d} \frac{x - a_i}{a_s - a_i}.$$

Now divide by the left-hand side and set $x = b_r$.