The 1/3-2/3 Conjecture for Coxeter groups

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Overview

1. Linear extensions and the 1/3-2/3 Conjecture
2. Weak order and convex sets
3. The fully commutative case
4. Generalized semiorders
5. A uniform lower bound
6. More examples

Linear extensions of posets

A **linear extension** of a poset P is an order-preserving bijection:

$$\lambda : P \to [n] = \{1, 2, \ldots, n\}.$$

Let $E(P)$ be the set of linear extensions of P and $e(P)$ be the number of linear extensions of P.

The 1/3-2/3 Conjecture

For $p, q \in P$, let $\delta(p,q)$ be the fraction of linear extensions such that $\lambda(p) < \lambda(q)$:

$$\delta(p,q) = \frac{|\lambda \in E(P) \mid \lambda(p) < \lambda(q)|}{e(P)}.$$

Conjecture (Kislicyn 1968; Fredman 1974; Linial 1984)

For any finite poset P that is not a total order, there exists p, q such that

$$\frac{1}{3} \leq \delta(p,q) \leq \frac{2}{3}.$$

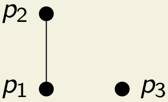
Define the **balance constant** to be

$$b(P) = \max_{p,q \in P} \min\{\delta(p,q), 1 - \delta(p,q)\}$$

Then the conjecture is saying that $b(P) \geq 1/3$.

Why 1/3?

Example



This poset has 3 linear extensions $p_1p_2p_3$, $p_1p_3p_2$, $p_3p_1p_2$, with $\delta(p_1, p_2) = 1$, $\delta(p_1, p_3) = 2/3$, $\delta(p_2, p_3) = 1/3$, thus $b(P) = 1/3$.

Known Results

It is nontrivial that $b(P)$ is uniformly bounded away from zero:

- $b(P) \geq 1/2e \approx 0.1839$ (Kahn and Linial 1991)
- $b(P) \geq 3/11 \approx 0.2727$ (Kahn and Saks 1984)
- $b(P) \geq (5 - \sqrt{5})/10 \approx 0.2764$ (Brightwell, Felsner, and Trotter 1995)

The full conjecture is known for:

- Width-2 posets
- Semiorders (unit interval orders)
- Height-2 posets
- Series parallel posets (N-free posets)

Coxeter groups and the weak order

A **Coxeter group** W is generated by a set of **simple reflections** $S = \{s_1, \ldots, s_r\}$ such that

$$W = \langle s_1, \ldots, s_r \mid s_i^2 = \mathrm{id},\ (s_i s_j)^{m_{i,j}} = \mathrm{id},\ \forall i, j \rangle$$

for some $m_{ij} \in \{2, 3, \ldots, \infty\}$.

Let $\ell(w)$ denote the **Coxeter length** of $w \in W$.

The (left) **weak order** $\leq_L$ is generated by

$$w \lessdot sw \quad \text{if } \ell(w) + 1 = \ell(sw).$$

Coxeter groups and the weak order

Let (W, S) be a **Coxeter system**.

The **reflections** T are the conjugates of simple reflections.

The (right) **inversion set** of $w \in W$ is

$$\mathrm{Inv}(w) := \{t \in T \mid \ell(wt) < \ell(w)\}.$$

Proposition

For $w, w' \in W$, $w \leq_L w'$ if and only if $\mathrm{Inv}(w) \subset \mathrm{Inv}(w')$.

It's not hard to see that $\ell(w) = |\mathrm{Inv}(w)|$.

Convex sets in Coxeter groups

Let H be the Hasse diagram of the weak order $\leq_L$.

A subset $C \subset W$ is called **convex** if for any $x, y \in C$, any shortest-length path between x and y remains in C.

Proposition (Tits 1974)

Any convex subset $C \subset W$ is of the form

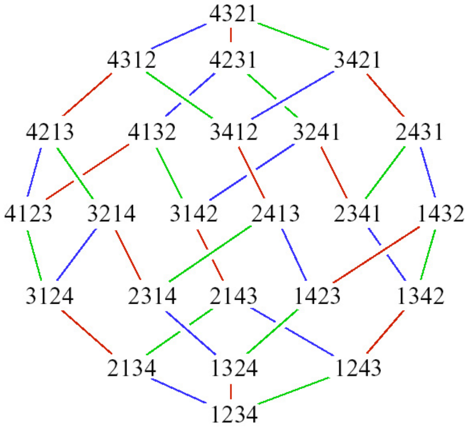
$$C = W_A^B := \{w \in W \mid A \subset \mathrm{Inv}(w) \subset B\}.$$

For our purposes, we usually translate a convex subset C by some element $w \in W$ to assume that $A = \emptyset$.

Example: the symmetric group $\mathfrak{S}_n$

The symmetric group $\mathfrak{S}_n$ is a Coxeter group generated by $S = \{(12), (23), \ldots (n{-}1\ n)\}$ with reflections $T = \{(ij) \mid 1 \leq i < j \leq n\}$. The reflection (ij) is an inversion of w if $w(i) > w(j)$.

$C = \{1234, 1243, 2134, 2143, 3142\}$ is convex and $C = W_{\emptyset}^{\{(13),(23),(24)\}}$.



Posets as convex sets in $\mathfrak{S}_n$

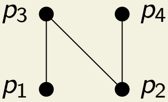
For a poset P on $\{p_1, \ldots, p_n\}$, we identify $E(P)$ as a set of permutations

$$C_P = \{\lambda \in E(P) \mid \lambda(1)\lambda(2)\cdots\lambda(n) \in \mathfrak{S}_n\}.$$

Proposition

The map $P \mapsto C_P$ gives a bijection between labeled poset on n elements and convex sets of $\mathfrak{S}_n$.

Example



corresponds to the convex set $\{1234, 2134, 2143, 1243, 2413\}$.

Balance constants for Coxeter groups

Recall the definition and we have

$$\delta(p_j, p_i) = \frac{|\{\lambda \in E(P) \mid \lambda(p_j) < \lambda(p_i)\}|}{e(P)} = \frac{|\{w \in C_P \mid (ij) \in \mathrm{Inv}(w)\}|}{e(P)}.$$

This suggests the following definition.
Let $C \subset W$ be a convex set and $t \in T$ be an inversion. Define

$$\delta(t) = \frac{|\{w \in C \mid t \in \mathrm{Inv}(w)\}|}{|C|}.$$

Define the **balance constant** to be

$$b(C) = \max_{t \in T} \min\{\delta(t), 1 - \delta(t)\}.$$

The 1/3-2/3 Conjecture for finite Coxeter groups

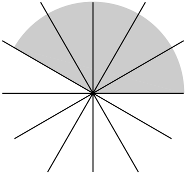
Remarkly, the natural generalization of the 1/3-2/3 Conjecture still seems to hold for finite Coxeter group.

Conjecture (Gaetz and G. 2020)

Let C be a convex set in a finite Coxeter group with $|C| > 1$, then $b(C) \geq 1/3$.

There are new equality cases in every type not coming from type A!

Convex sets in finite Coxeter groups are exactly convex sets of regions in the corresponding Coxeter arrangement. The conjecture says that there is a hyperplane that cuts the convex set into roughly equal parts.



Fully commutative elements in Weyl groups

Definition (Stembridge 1996)

An element $w \in W$ is **fully commutative** if no reduced word of w contains a consecutive substring $\underbrace{s_i s_j s_i \cdots}_{m_{ij}}$ with $m_{ij} > 2$.

Recall that we have relations $(s_i s_j)^{m_{ij}} = \mathrm{id}$.

Example

In $\mathfrak{S}_4$, $s_2 s_1 s_3 s_2 = 3412$ is fully commutative but $s_2 s_1 s_2 s_3 = 3241$ is not.

Fully commutative elements and width-2 poset

Proposition

A permutation $w \in \mathfrak{S}_n$ is fully commutative if and only if w avoids 321.

Definition

A poset has **width** k if its longest antichain has size k.

Proposition

Under the bijection between (naturally labeled) posets and convex sets in $\mathfrak{S}_n$, width-2 posets P correspond to convex sets $C_P = [\mathrm{id}, w]_L$ where w is fully commutative.

The 1/3-2/3 Conjecture is known for width-2 poset. We strengthen this result to other Coxeter group.

Fully commutative elements in Coxeter group

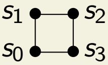
Theorem (Gaetz and G. 2020)

Let $w \neq \mathrm{id} \in W$ be a fully commutative element in any Coxeter group W whose Coxeter diagram is acyclic, then $b([\mathrm{id}, w]_L) \geq 1/3$.

In particular, this theorem applies to all finite Coxeter groups.

Example

In the affine Weyl group $\widetilde{A_3}$, $w = s_1 s_3 s_0 s_2$ is fully commutative, but $b([id, w]_L) = 2/7$.



Heaps of fully commutative elements

The proof of this theorem uses the theory of heaps.

Definition (Stembridge 1996)

Let $\boldsymbol{s} = s_{i_1} \cdots s_{i_\ell}$ be a reduced word of w. The **Heap poset** $H_{\boldsymbol{s}}$ is the partial order $([\ell], \preceq)$, which is the transitive closure of

$$j \preceq k \text{ if } j \leq k \text{ and } (s_{i_j} s_{i_k})^2 \neq \mathrm{id}.$$

In the case where w is fully commutative, the Heap poset does not depend on the choice of a reduced word. We write the Heap poset as H_w.

Proposition (Stembridge 1996)

Let $w \in W$ be fully commutative. Then $[\mathrm{id}, w]_L \simeq J(H_w)$.

Here $J(H_w)$ is the lattice of order ideals of H_w.

Balance constants for order ideals

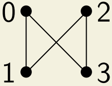
Let Q be a poset and $J(Q)$ be the order ideals of Q. Define

$$\delta^{\text{ideal}}(x,Q) = \frac{|\{J \in J(Q) \mid x \in J\}|}{|J(Q)|},$$
$$b(Q) = \max_{x\in Q} \min\{\delta^{\text{ideal}}(x,Q), 1-\delta^{\text{ideal}}(x,Q)\}.$$

For fully commutative w, we have $b([\mathrm{id},w]_L) = b(H_w)$.

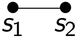
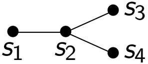
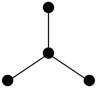
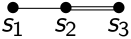
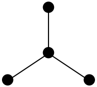
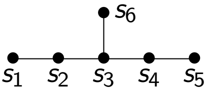
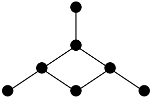
Example: non-example

Consider the following poset $Q = H_w$, $w = s_1s_3s_0s_2 \in W(\widetilde{A_3})$.
We see that $|J(Q)| = 7$ and $b(Q) = 2/7$.



Equality cases

These intervals $C = [e, w]_L$ give equality cases $b(C) = 1/3$:

type	diagram	w	H_w
A_2		$s_1 s_2$	
D_4		$s_4 s_2 s_3 s_1$	
B_3		$s_3 s_2 s_3 s_1$	
E_6		$s_6 s_3 s_2 s_4 s_1 s_3 s_5$	

Semiorders

Definition

A poset P is a **semiorder** (**unit-interval order**) if there exists a function $f : P \to \mathbb{R}$ such that $p < q$ if and only if $f(q) - f(p) > 1$.

Theorem (Brightwell 1989)

If P is a semiorder which is not a chain, then $b(P) \geq 1/3$.

We will be generalizing this result to finite Weyl groups.

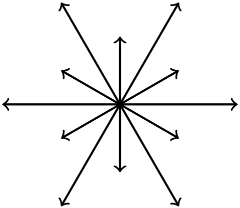
Root systems and Weyl groups

Definition (Root system)

Let $E = \mathbb{R}^n$. A **root system** $\Phi \subset E$ is a finite set of vectors, such that

- Φ spans E;
- for $\alpha \in \Phi$, $k\alpha \in \Phi$ iff $k \in \{\pm 1\}$;
- for $\alpha, \beta \in \Phi$, $2\langle \alpha, \beta \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}$;
- for $\alpha, \beta \in \Phi$,

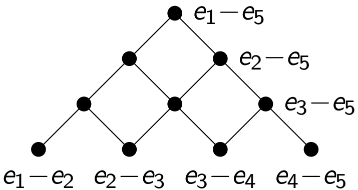
$$\sigma_\alpha(\beta) := \beta - \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha \in \Phi.$$



Root systems and Weyl groups

Let $\Phi \subset E$ be a root system.

- The **Weyl group** $W(\Phi) \subset \mathrm{GL}(E)$ is generated by reflections $\{\sigma_\alpha \mid \alpha \in \Phi\}$. It's a Coxeter group.
- We can partition Φ into **positive roots** Φ^+ and negative roots Φ^-.
- Given $\Phi = \Phi^+ \sqcup \Phi^-$, there is a unique choice of **simple roots** $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset \Phi^+$ such that each $\alpha \in \Phi^+$ can be written as an integral linear combination of Δ.
- There is a partial order on Φ^+ given by $\alpha \leq \beta$ if $\beta - \alpha$ is a nonnegative linear combinationn of Δ.
- The minimal elements of Φ^+ are the simple roots and the unique maximum of Φ^+ is the **highest root** ξ.



Generalized semiorders

Let W be a finite crystallographic Weyl group. Recall

$$W_A^B = \{w \in W \mid A \subseteq Inv(w) \subseteq B\}.$$

Definition (Gaetz and G. 2020)

A convex set $C = W_\emptyset^B$ is a **generalized semiorder** if B is an order ideal of the root poset Φ^+.

It recovers the definition of a semiorder in type A. Recall that a semiorder on P is defined by $p < q$ if $f(q) - f(p) > 1$ for some $f : P \to \mathbb{R}$.
Say $P = \{p_1, \ldots, p_n\}$ where $f(p_1) < \cdots < f(p_n)$. Then

$$\{e_i - e_j \mid i < j,\ f(j) - f(i) \leq 1\}$$

is an order ideal of Φ^+ in type A_{n-1}.

Generalized semiorders

Theorem (Gaetz and G. 2020)

Let C be a non-singleton generalized semiorder, then $b(C) \geq 1/3$.

This theorem relies on the following purely root-theoretic fact.

Lemma (Gaetz and G. 2020)

Let $J \subset \Phi^+$ be a nonempty order ideal. Then there exists a simple root $\alpha \in J$ such that we cannot find $\beta_1 \neq \beta_2 \in J$ with $s_\alpha\beta_1, s_\alpha\beta_2 \in \Phi^+ \setminus J$.

Our proof is very bad. It's very technical and type dependent.

A uniform lower bound

It is nontrivial that $b(C)$ is bounded away from 0.
Let W be a finite crystallographic Weyl group of rank r, highest root ξ, and fundamental coweights $\omega_1^\vee,\ldots,\omega_r^\vee$. Define

$$\begin{aligned} m_0 &= \min_i \langle \omega_i^\vee, \xi \rangle, \\ m_1 &= \max_i \langle \omega_i^\vee, \xi \rangle, \\ m &= r/m_0 + 1/m_1 - \mathrm{ht}(\xi)/m_0 m_1. \end{aligned}$$

Theorem (Gaetz and G. 2020)

Let C be a non-singletone convex set in W, then

$$b(C) \geq \frac{1}{2e^{mm_1}} \geq \frac{1}{2e^{12}}.$$

A uniform lower bound

Theorem (Gaetz and G. 2020)

Let C be a non-singletone convex set in W, then

$$b(C) \geq \frac{1}{2e^{mm_1}}.$$

Type	m_0	m_1	$\mathrm{ht}(\Phi)$	m	mm_1	$b(C) \geq$
A_r	1	1	r	1	1	$1/2e$
B_r	1	2	$2r-1$	1	2	$1/2e^2, 1/2e$
C_r	1	2	$2r-1$	1	2	$1/2e^2, 1/2e$
D_r	1	2	$2r-3$	2	4	$1/2e^4$
E_6	1	3	11	$8/3$	8	$1/2e^8$
E_7	1	4	17	3	12	$1/2e^{12}$
E_8	2	6	29	$7/4$	$21/2$	$1/2e^{10.5}$
F_4	2	4	11	$7/8$	$7/2$	$1/2e^{3.5}$
G_2	2	3	5	$1/2$	$5/2$	$1/2e^{2.5}, 1/3$

Generalized order polytope

Let $W \subset \mathrm{GL}(E)$ be a finite crystallographic Weyl group. The **fundamental alcove** is

$$Q_{\mathrm{id}} := \{v \in E \mid \langle v, \alpha \rangle \geq 0 \ \forall \alpha \in \Delta, \langle v, \xi \rangle \leq 1\}.$$

For each $w \in W$, its corresponding alcove is

$$Q_w := w^{-1} Q_{\mathrm{id}}.$$

Definition (Gaetz and G. 2020)

For a convex set $C \subset W$, the **generalized order polytope** is

$$\mathcal{O}(C) := \bigcup_{w \in C} Q_w.$$

Generalized order polytope

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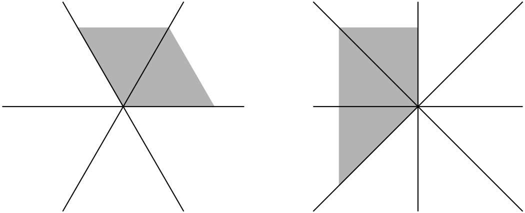


These are special cases of **alcoved polytopes** (Lam and Postnikov 2005).

A geometric approach

For a reflection $t \in T$, let H_t be its corresponding hyperplane.
The hyperplane H_t cuts $\mathcal{O}(C)$ into two parts $\mathcal{O}(C)_t^+$ and $\mathcal{O}(C)_t^-$.
The goal is to show that there exists $t \in T$ such that both $\mathcal{O}(C)_t^+$ and $\mathcal{O}(C)_t^-$ have volumes at least $\epsilon \mathrm{Vol}(\mathcal{O}(C))$.

Lemma

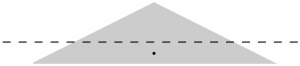
Let C be a non-singleton convex set of a finite crystallographic Weyl group of rank r and let o_C be the centroid of $\mathcal{O}(C)$. Then there exists $t \in \Phi^+$ with alcoves on both sides of H_t, such that

$$|\langle o_C, t\rangle| \leq \frac{m}{r+1}.$$

In other words, there is a hyperplane H_t that is close to the centroid.

A geometric approach

We now have a hyperplane H_t close to the centroid o_C.



Proposition

Let $Q \subset \mathbb{R}^n$ be a full-dimensional compact convex body with centroid c_Q. Let $m \geq 1$ and $v \in \mathbb{R}^n$ such that $\langle v, c_Q \rangle \geq -m/(n+1)$, $\min_{x \in Q}\langle v, x\rangle \leq -1$, $\max_{x \in Q}\langle v, x\rangle \geq 1$, then

$$\frac{\mathrm{Vol}(Q_v^+)}{\mathrm{Vol}(Q)} \geq \frac{1}{2e^m}.$$

This proposition says that we can obtain a bound if the centroid c_Q is close to the hyperplane H_v, and the hyperplane is relatively in the middle of Q.

A geometric approach

The proof generalizes the argument by Kahn and Linial, which uses the following immediate corollary of Brunn-Minkowski.

Lemma (Brunn-Minkowski)

The function $\lambda \mapsto \mathrm{Vol}(Q_v^\lambda)^{\frac{1}{n-1}}$ *is concave on* $\mathbb{R}$, *where* $Q \subset \mathbb{R}^n$ *is a convex body and* $Q_v^\lambda := \{x \in Q \mid \langle v, x\rangle = \lambda\}$.

The proof of the proposition explicitly constructs the convex body Q that minimizes $\mathrm{Vol}(Q_v^+)/\mathrm{Vol}(Q)$, which is a double cone.

A twist in type B_n

We can achieve a better bound in type B_n, $1/2e$ instead of $1/2e^2$, with an alternative definition of generalized order polytopes.
Define the alcoves

$$Q_{\mathrm{id}}^{\eta} := \{v \in E \mid \langle v, \alpha \rangle \geq 0 \ \forall \alpha \in \Delta, \langle v, \eta \rangle \leq 1\}$$

and $Q_w^{\eta} := w^{-1} Q_{\mathrm{id}}^{\eta}$, where η is the highest short root.

Definition (Gaetz and G. 2020)

For a convex set $C \subset W$, the **short-root order polytope** in type B_n is $\mathcal{O}(C) := \bigcup_{w \in C} Q_w^{\eta}$.

The standard realization of the type B_n root system

- $\Phi(B_n) = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\}$,
- simple roots $\Delta = \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_n\}$,
- highest root $\xi = e_1 + e_2$, highest short root $\eta = e_1$.

More examples

Recall that $b(C) \geq \epsilon$ for finite Weyl groups and an absolute constant ϵ.

Example: no universal bound for Coxeter groups

- Let W_n be the Coxeter group whose Dynkin diagram is a complete graph (with $m_{ij} \geq 3$) on n generators $\{s_1, \ldots, s_n\}$, where $n \geq 3$.
- Consider the convex set

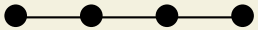
$$C = \mathrm{conv}(s_1, s_2, \ldots, s_n) = \{\mathrm{id}, s_1, s_2, \ldots, s_n\}.$$

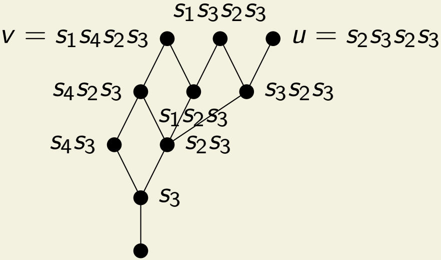
- Then $\delta(s_i) = 1/(n+1)$ and $b(C) = 1/(n+1)$.
- So there are no lower bounds for $b(C)$ in arbitrary Coxeter groups.

More examples

In the original 1/3-2/3 Conjecture on posets, it is believed (Brightwell 1999) that width-2 posets come closest to violating the conjecture. This heuristics is not true for fully commutative elements.

Example: heuristic on fully commutative elements

Let W be  with $m_{12} \geq 4$, $m_{23} \geq 7$, $m_{34} \geq 4$.
Since W is acyclic, our theorem says $b([\mathrm{id}, w]_L) \geq 1/3$ for $w \neq \mathrm{id}$ fully commutative.
Consider $C = \mathrm{conv}(\mathrm{id}, u, v)$. We have $b(C) = 3/10 < 1/3$.



Thanks

Thank you for listening!