Completing and extending shellings of vertex decomposable complexes

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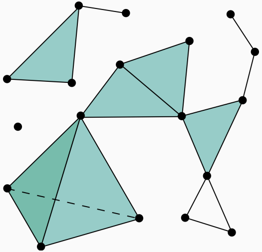
Simplicial complexes

A **simplicial complex** Δ on a set V is a collection of subsets of V such that if $\sigma \in \Delta$ and $\tau \subset \sigma$, then $\tau \in \Delta$.

- The *dimension* of Δ is the size of a maximal element minus one.
- Δ is *pure* if all maximal elements (called facets) have the same cardinality.



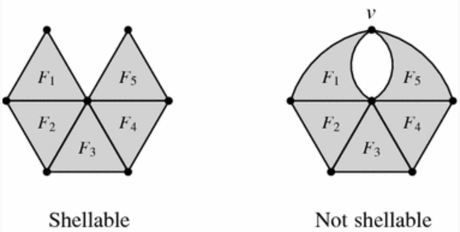
- Combinatorial structure allows for computing algebraic invariants, can be studied with tools from commutative algebra via their Stanley-Reisner rings.
- Example: $< 123, 456 >= \{\emptyset, 1, 2, 3, 4, 5, 6, 12, 13, 23, 45, 46, 56, 123, 456\}$

Shellable complexes

A pure d-dim simplicial complex Δ is **shellable** if there exists an ordering of its facets $F_1, \ldots, F_s$ such that for all k the complex

$$\left\langle \bigcup_{i=1}^{k-1} F_i \right\rangle \cap \langle F_k \rangle$$

is pure of dimension $d - 1$.



Motivations for shellability

- Shellable complexes have the homotopy type of a wedge of spheres (or are contractible).
- If Δ is shellable then its Stanley-Reisner ring R/I_Δ is Cohen-Macaulay.
- Examples:
  - Boundaries of simplicial polytopes [Brugesser-Mani],
  - Independence complexes of matroids [Björner],
  - Skeleta of shellable complexes, e.g. $\Delta_{n-1}^{(k)}$, the k-skeleton of a simplex on $[n]$.
- Recently Goaoc, Paták, Patakova, Tancer, Wagner proved that for $d \geq 2$ deciding if a given d-dimensional complex on n facets is shellable is NP-complete.
- Easy to get h-vector with a shelling order.

Extendably shellable complexes

A shellable complex Δ is said to be **extendably shellable** (ES) if any shelling of a subcomplex can be extended to a shelling of Δ.

- Any 2-dim triangulated sphere is ES [Danaraj-Klee].
- All rank 3 matroids are ES [Björner-Eriksson].
- Any d-sphere with $d + 3$ vertices is ES [Kleinschmidt].
- Some 'nicely behaved' shellable complexes are not ES (e.g. certain simplicial 3-spheres [Ziegler]).

Simon's Conjecture

The motivation for much of our work will be the following question posed by Simon.

Conjecture (Simon's Conjecture)

The complex $\Delta_{n-1}^{(k)}$ is extendably shellable.

- $k = 2$ case follows from by considering the uniform matroid of rank 3.
- Bigdeli, Yazdan Pour, and Zaare-Nahandi have established the $k \geq n - 3$ cases (strengthened by Culbertson, Dochtermann, Guralnik, Stiller).

Definition

A pure d-dimensional complex Δ on n vertices is **shelling completable** if there exists a shelling $F_1, F_2, \ldots, F_s$ of Δ that can be taken as the initial sequence of some shelling of $\Delta_{n-1}^{(d)}$.

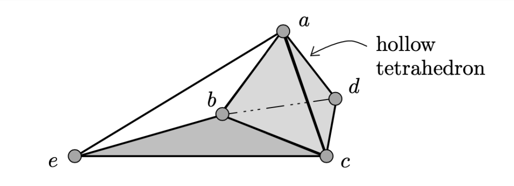
- If Δ is shelling completable then *any* shelling of Δ can be completed.
- Simon's conjecture: any pure shellable complex is shelling completable.
- We prove that a nice subclass is shelling completable.

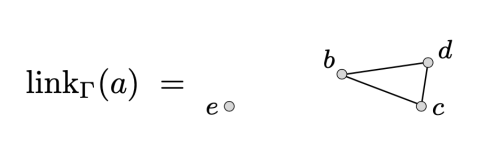
Link and Deletion

For a simplicial complex Δ on ground set V and face $F \in \Delta$, we have

$$\mathrm{lk}_\Delta(F) := \{G \in \Delta : G \cap F = \emptyset, G \cup F \in \Delta\},$$

$$\mathrm{del}_\Delta(F) := \{G \in \Delta : F \not\subseteq G\}.$$





Helpful fact: the link of a shellable complex is shellable.

Vertex decomposable complexes

Definition

A simplicial complex Δ is **vertex decomposable** (VD) if Δ is a simplex, or Δ contains a vertex v (decomposing vertex) such that

1. both $\mathrm{del}_\Delta(v)$ and $\mathrm{lk}_\Delta(v)$ are vertex decomposable, and
2. any facet of $\mathrm{del}_\Delta(v)$ is a facet of Δ.

- Introduced by Provan-Billera in their study of the Hirsh conjecture.
- Vertex decomposable complexes are shellable.
- The k-skeleta of a simplex is VD.
- Shifted complexes are VD.
- (Provan-Billera) Matroid complexes are VD, and any vertex works!
- Bergman complex of matroids are not.

Vertex decomposable implies shelling completable

With Coleman, Dochtermann and Oh we prove

Theorem [CDGO, '20]

Suppose Δ is a d-dimensional vertex decomposable simplicial complex on ground set V. Then there exists a linear order on V such that if F is the revlex smallest $(d+1)$-subset of V not contained in Δ, then the complex $\langle \Delta \cup \{F\} \rangle$ is vertex decomposable.

From this we conclude:

Corollary

Vertex decomposable complexes are shelling completable.

Example of completing and idea of proof

Suppose Δ is a VD complex. The basic idea is to consider a decomposing vertex and use induction. If the deletion has a missing facet, induction gives the desired ordering needed to extend; otherwise consider the link.

1. For example let $\Delta = \{1234, 1235, 1245, 1345, 2345, 1236, 1246, 1256, 2356, 1237, 2347\}$, and note that 7 is decomposing.
   Here $del_{\Delta}(7) = \{1234, 1235, 1245, 1345, 2345, 1236, 1246, 1256, 2356\}$ has decomposing order $\{1, 2, 3, 4, 5, 6\}$ and we can add the facet 1346.

2. Next let $\Delta' = \{1234, 1235, 1245, 1345, 2345, 1236, 1246, 1256, 2356\}$, and note that 6 is decomposing.
   Now we have $del_{\Delta'}(6) = \{1234, 1235, 1245, 1345, 2345\}$ is 'full', so we consider $lk_{\Delta'}(6) = \{123, 124, 125, 235\}$ which has decomposing order $\{1, 2, 3, 4, 5\}$ and extendable by 134. We then add the facet 1346 to the complex.

Applications to matroids

A pure simplicial complex Δ is (the independence complex of) a **matroid** if its set of facets satisfies the following exchange property: If F and G are facets of Δ then for any $x \in F \backslash G$ there exists some $y \in G \backslash F$ such that $(F \backslash \{x\}) \cup \{y\}$ is a facet of M.

- Matroids are VD, hence shelling completable.
- What facet(s) can be added to maintain VD?

Proposition (CDGO, '20)

Let Δ be a rank d matroid and suppose $v_1, v_2, \ldots, v_n$ is any ordering of its ground set such that $\{v_1, v_2, \ldots, v_d\} \in \Delta$. If F is the revlex smallest d-subset missing from Δ then the complex generated by $\Delta \cup \{F\}$ is vertex decomposable.

- Related question: does there exist a d-subset $F \subset V$ such that $\Delta \cup \{F\}$ is again a *matroid*? By results of Kahn and Truemper this holds if and only if F is a circuit-hyperplane.

Complexes with cofew vertices

Culbertson, Dochtermann, Guralnik, and Stiller have shown that if Δ is a shellable d-dim complex on at most $d+3$ vertices then Δ is extendably shellable. Similarly we prove:

Theorem (CDGO, '20)

Suppose Δ is a shellable d-dimensional simplicial complex on at most $d+3$ vertices. Then Δ is vertex decomposable.

- Hence for such complexes the concepts of shellable, extendably shellable, shelling completable, and vertex decomposable are all equivalent.
- Theorem is tight in the sense that there exist 2-dimensional complexes on 6 vertices that are shellable but not vertex decomposable.

Further directions: k-decomposable complexes

Definition

A complex Δ is k-**decomposable** if it is a simplex or contains a face F of dim $\leq k$ such that

1. both $\mathrm{del}_\Delta(F)$ and $\mathrm{lk}_\Delta(F)$ are k-decomposable, and
2. any facet of $\mathrm{del}_\Delta(F)$ is a facet of Δ.

- For any pure d-dim complex, we have

$$0\text{-dec } (= \text{VD}) \subseteq 1\text{-dec} \subseteq 2\text{-dec} \cdots \subseteq d\text{-dec } (= \text{Shellable}).$$

- Question: Can one extend a k-dec complex by one facet, so it is still k-dec?
- When $k = 0$ this is our result for VD, when $k = d$ this is Simon's conjecture!
- Question: Is it true that 1-dec complexes are shelling completable? (Work in Progress.)