Coefficientwise total positivity of some matrices defined by linear recurrences

Tomack Gilmore

Joint work with X. Chen, B. Deb, A. Dyachenko, and A. D. Sokal

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Ramat-Gan, Israel (and online)

The Eulerian Triangle

$$\left(\left\langle {n \atop k} \right\rangle\right)_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 4 & 1 & 0 & 0 & \cdots \\ 1 & 11 & 11 & 1 & 0 & \cdots \\ 1 & 26 & 66 & 26 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The Eulerian triangle (OEIS: A008292)

- Entries satisfy the **linear recurrence**:

$$\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle$$

  for $n \geq 1$ with initial condition $\left\langle {0 \atop k} \right\rangle = \delta_{0k}$.
- $\left\langle {n \atop k} \right\rangle$ counts **permutations** of $[n] = \{1, 2, \ldots, n\}$ with k **descents** (or k **ascents**);

$$\left(\left\langle {n \atop k} \right\rangle^{\text{clean}} \right)_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 4 & 1 & 0 & 0 & \cdots \\ 1 & 11 & 11 & 1 & 0 & \cdots \\ 1 & 26 & 66 & 26 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The "clean" Eulerian triangle

- Entries satisfy the **linear recurrence**:

$$\left\langle {n \atop k} \right\rangle^{\text{clean}} = (n-k+1) \left\langle {n-1 \atop k-1} \right\rangle^{\text{clean}} + (k+1) \left\langle {n-1 \atop k} \right\rangle^{\text{clean}}$$

  for $n \geq 1$ with initial condition $\left\langle {0 \atop k} \right\rangle = \delta_{0k}$.

- $\left\langle {n \atop k} \right\rangle^{\text{clean}}$ counts **permutations** of $[n+1]$ with k **descents**;

$$\left(\left\langle {n \atop k} \right\rangle^{\text{clean}}_{n,k\geq 0} \right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 4 & 1 & 0 & 0 & \cdots \\ 1 & 11 & 11 & 1 & 0 & \cdots \\ 1 & 26 & 66 & 26 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Conjecture (Brenti '96)

The clean Eulerian triangle is ***totally positive****.*

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The clean Eulerian triangle is ***totally positive****.*

Definition

A (finite or infinite) matrix with entries belonging to $\mathbb{R}$ is **totally positive** if **all of its minors** are **nonnegative**.

The matrix $\mathbf{T}(a, c, d, e, f, g)$

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Let $\mathbf{T}(a, c, d, e, f, g) = (\mathbf{T}(n, k))_{n,k\geq 0}$ be the matrix with entries that satisfy the linear recurrence

$$T(n, k) = [\mathbf{a}(n-k) + \mathbf{c}]T(n-1, k-1) + (\mathbf{d}k + \mathbf{e})T(n-1, k) + [\mathbf{f}(n-2) + \mathbf{g}]T(n-2, k-1)$$

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for $n \geq 1$ with initial conditions $\mathbf{T}(0, k) = \delta_{0k}$ and $\mathbf{T}(-1, k) = 0$.

- Here a, c, d, e, f, g are treated as **algebraic indeterminants**.
- The entries of $\mathbf{T}(a, c, d, e, f, g)$ belong to the **polynomial ring** $\mathbb{Z}[a, c, d, e]$.
- $\mathbf{T}(a, c, d, e, f, g)$ defines a **family of integer matrices**.

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- $\mathbf{T}(0, 1, 0, 1, 0, 1)$ is the **Delannoy triangle**.

$$\begin{pmatrix} 1 & 0 & \cdots \\ e & c & \cdots \\ e^2 & ae + c(d+2e) + g & \cdots \\ e^3 & ae(d+3e) + c\left(d^2+3de+3e^2\right) + dg + ef + 2eg & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

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Definition

Equip the polynomial ring $\mathbb{R}[\mathbf{x}]$ with the **coefficientwise partial order**. Then $P \in \mathbb{R}[\mathbf{x}]$ is **nonnegative** in case P is a polynomial with **nonnegative coefficients** (written $P \succeq 0$).

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Definition

A (finite or infinite) matrix with entries in $\mathbb{R}[\mathbf{x}]$ is **coefficientwise totally positive** if all of its minors are **nonnegative polynomials**.

$$\begin{pmatrix} 1 & 0 & \cdots \\ e & c & \cdots \\ e^2 & ae + c(d+2e) + g & \cdots \\ e^3 & ae(d+3e) + c\left(d^2 + 3de + 3e^2\right) + dg + ef + 2eg & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Conjecture (X. Chen, B. Deb, A. Dyachenko, TG, A. D. Sokal '21)

The matrix $\mathbf{T}(a, c, d, e, f, g)$ *is* **coefficientwise totally positive**.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ e & c & 0 & 0 & \cdots \\ e^2 & ae + 2ce & c^2 & 0 & \cdots \\ e^3 & 3ae^2 + 3ce^2 & a^2e + 3ace + 3c^2e & c^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Theorem (X. Chen, B. Deb, A. Dyachenko, TG, A. D. Sokal '21)

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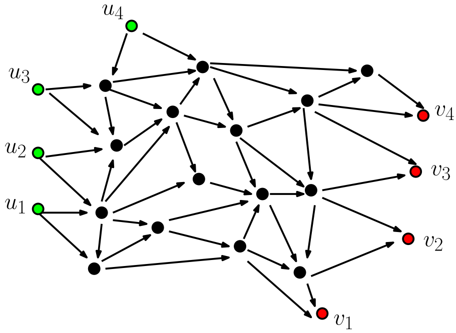
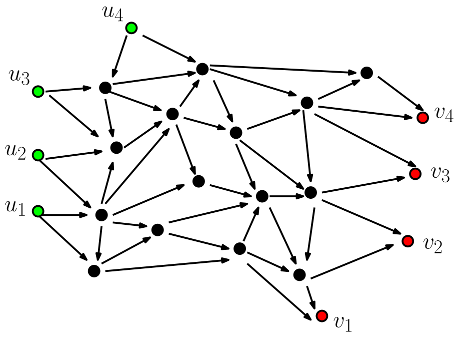
From now on let $\mathbf{T}(a, c, e)$ denote $\mathbf{T}(a, c, 0, e, 0, 0)$.

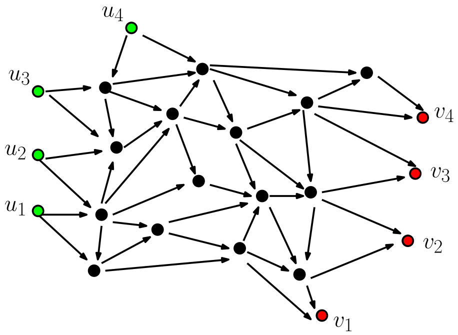
Planar networks



A **locally finite acyclic digraph** D with sources $U = \{u_1, u_2, \ldots, u_n\}$ and sinks $V = \{v_1, v_2, \ldots, v_k\}$.



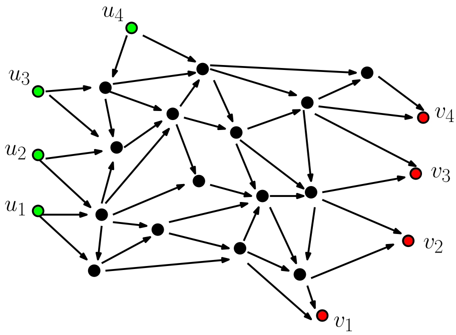
- Each edge $e \in D$ is assigned a weight w_e belonging to some **commutative ring**.
- **Weight of a path** from u_n to v_k: $w(\mathcal{P}) = \prod_{\mathcal{P}: u_n \to v_k} w_e$;
- **Path matrix**: $\mathbf{P}_D = (P(u_n \to v_k))_{n,k \geq 1}$ where $P(u_n \to v_k) = \sum_{\mathcal{P}: u_n \to v_k} w(\mathcal{P})$.



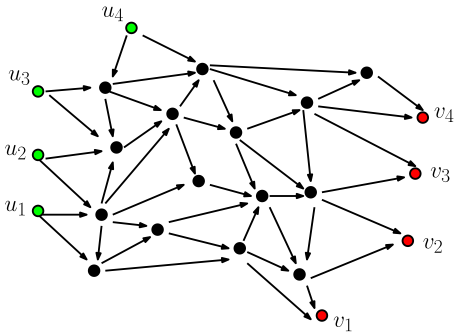
Lemma (KMLGV Lemma)

$$\det(\mathbf{P}_D) = \sum_{(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_n):U\to V} \operatorname{sgn}(\sigma(\mathcal{P})) \prod_{i=1}^{n} w(\mathcal{P}_i)$$

$\det(\mathbf{P}_D)$ gives the **signed sum** over **weighted families of nonintersecting paths**.



U and V are **fully compatible** if for **any subset** of **sources** $u_{n_1}, u_{n_2}, \dots, u_{n_r}$ (with $n_1 < n_2 < \cdots < n_r$) and **sinks** $v_{k_1}, v_{k_2}, \dots, v_{k_r}$ (with $k_1 < k_2 < \cdots < v_{k_r}$), the **only** permutation $\sigma \in \mathfrak{S}_r$ mapping u_{n_i} to $v_{k_{\sigma(i)}}$ giving rise to a **nonempty family of nonintersecting paths** is $\sigma = 1$.



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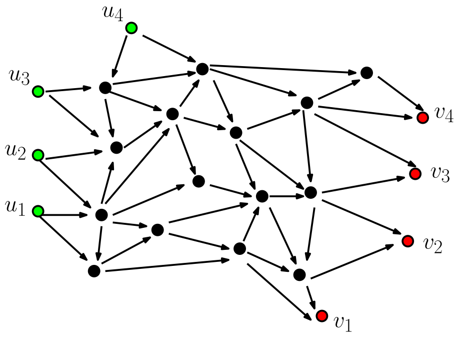
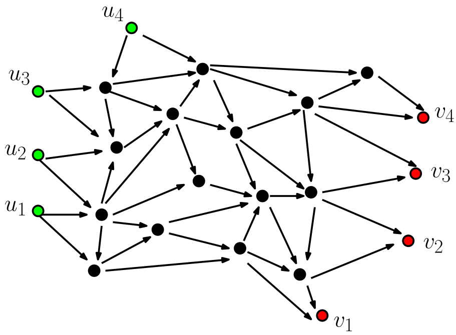
A **planar network** is a **locally finite acyclic digraph** with **fully compatible sources and sinks**

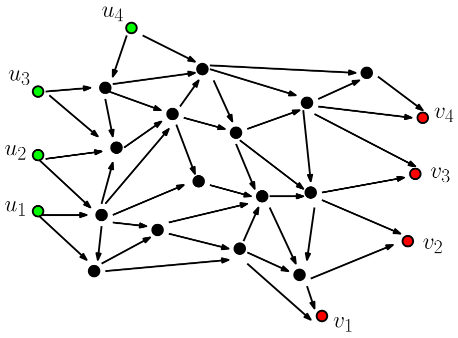


u_4
u_3
u_2
u_1
v_4
v_3
v_2
v_1



Total positivity and LGV lemma

- U and V **fully compatible** $\rightarrow$ every minor of $\mathbf{P}_D$ is a sum over weighted **families of nonintersecting paths**.



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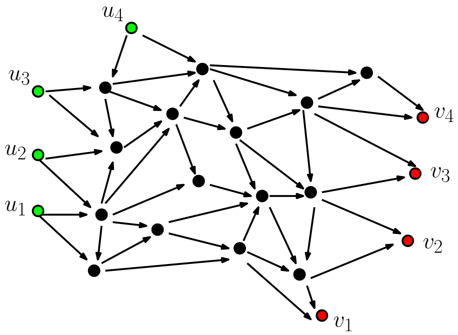
- U and V **fully compatible** $\rightarrow$ every minor of $\mathbf{P}_D$ is a sum over weighted **families of nonintersecting paths**.
- If $w_e \in \mathbb{R}, w_e \geq 0 \rightarrow \mathbf{P}_D$ is **totally positive**;



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- U and V **fully compatible** $\rightarrow$ every minor of $\mathbf{P}_D$ is a sum over weighted **families of nonintersecting paths**.
- If $w_e \in \mathbb{R}, w_e \geq 0 \rightarrow \mathbf{P}_D$ is **totally positive**;
- If $w_e \in \mathbb{R}[\mathbf{x}], w_e \succeq 0 \rightarrow \mathbf{P}_D$ is **coefficientwise totally positive** in $\mathbf{x}$.

The matrix $\mathbf{T}(a, c, e)$

Theorem (X. Chen, B. Deb, A. Dyachenko, TG, A. D. Sokal '21)

The matrix $\mathbf{T}(a, c, e) = (T(n, k))_{n,k \geq 0}$ *with entries satisfying*

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Combinatorial interpretation

The entries of $\mathbf{T}(\mathbf{a},\mathbf{c},\mathbf{e})$ satisfy

$$T(n,k) = \sum_{\pi\in\Pi_{n+1,n+1-k}} \prod_{i=2}^{n+1} w_\pi(i)$$

where

$$w_\pi = \begin{cases} \mathbf{c} & \text{if smallest}(\pi,i) = 1, \\ \mathbf{e} & \text{if smallest}(\pi,i) = i, \\ \mathbf{a} & \text{if smallest}(\pi,i) \neq i, 1. \end{cases}$$

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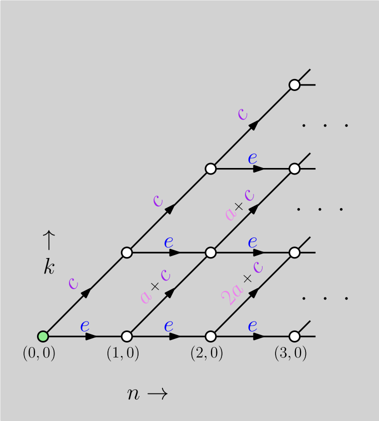

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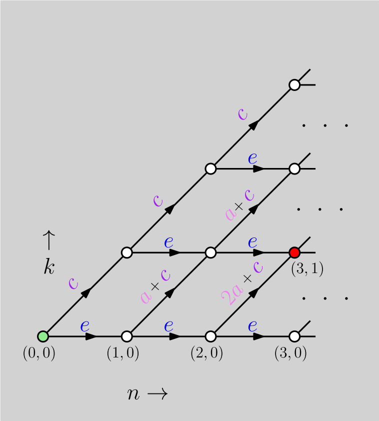

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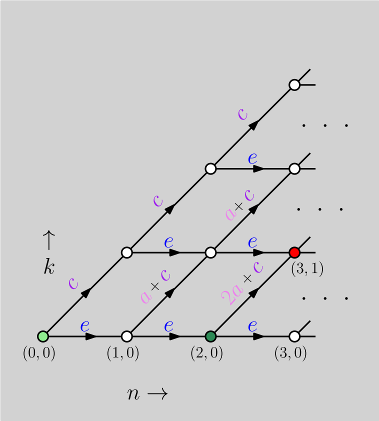

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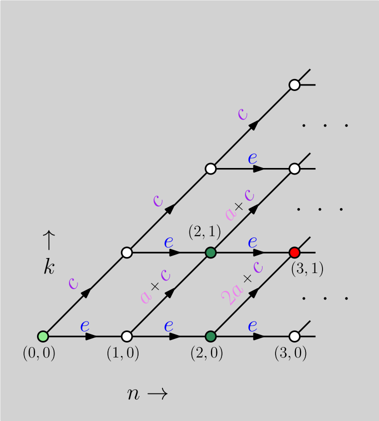

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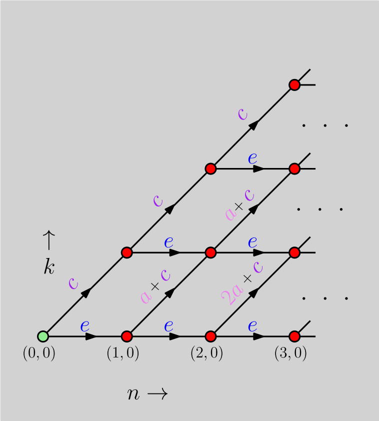

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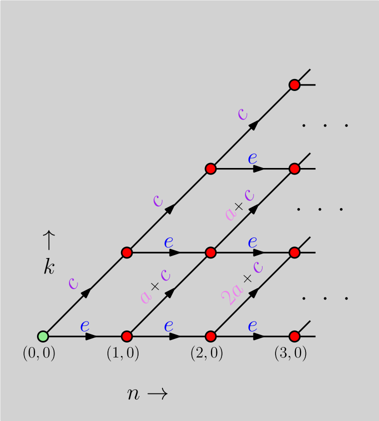

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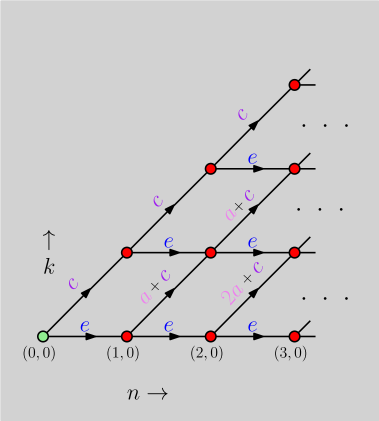

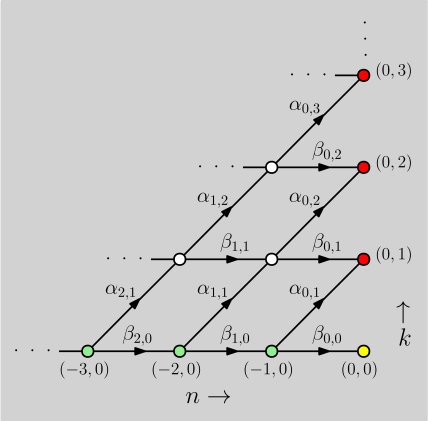

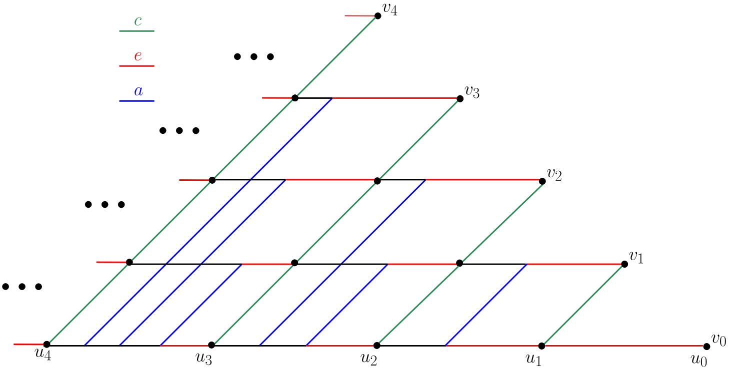

c
e
a
v4
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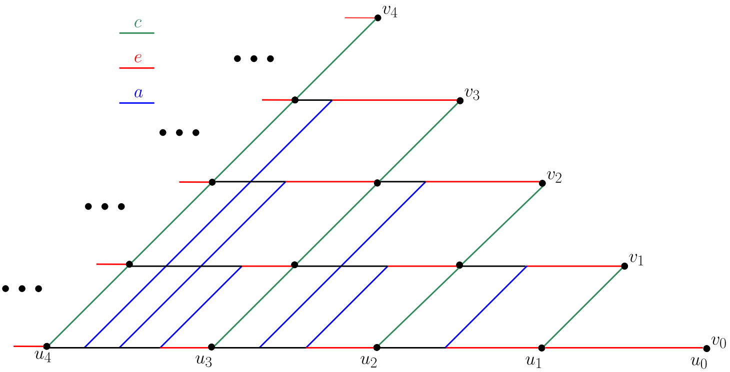
v2
v1
v0
u4
u3
u2
u1
u0



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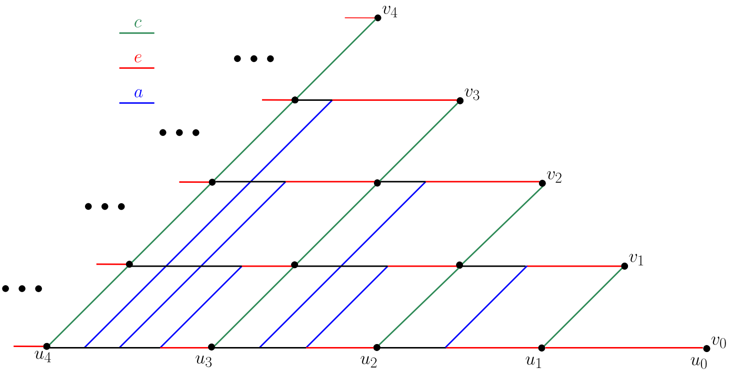
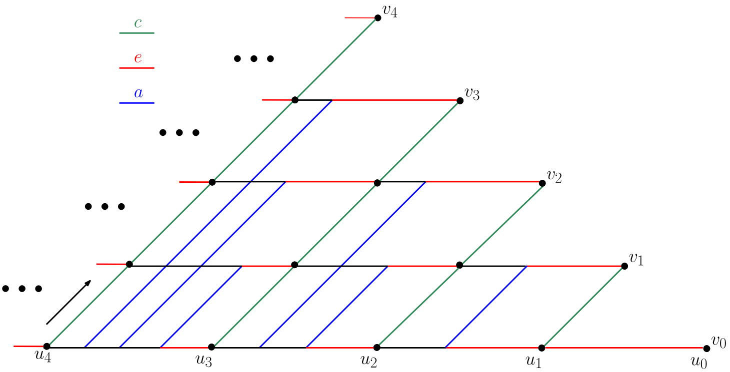


The entries of $\mathbf{T}(a, c, e)$ satisfy the **alternative recurrence**:

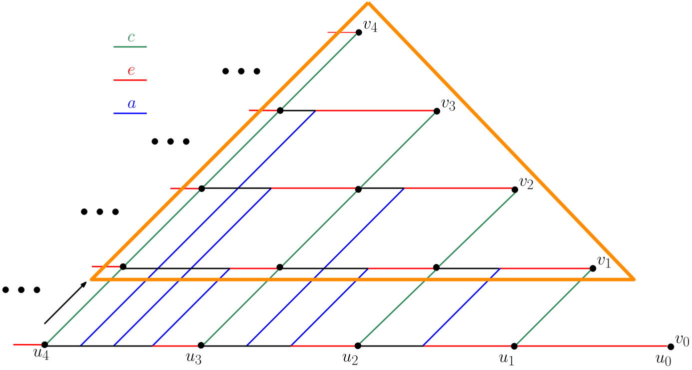
$$T(n,k) = c\,T(n-1,k-1) + \sum_{m=0}^{n-1} \binom{n-1}{m} a^m e\,T(n-1-m,k-m)$$

for $n \geq 1$, where $T(n,k) = 0$ if $n < 0$ or $k < 0$.



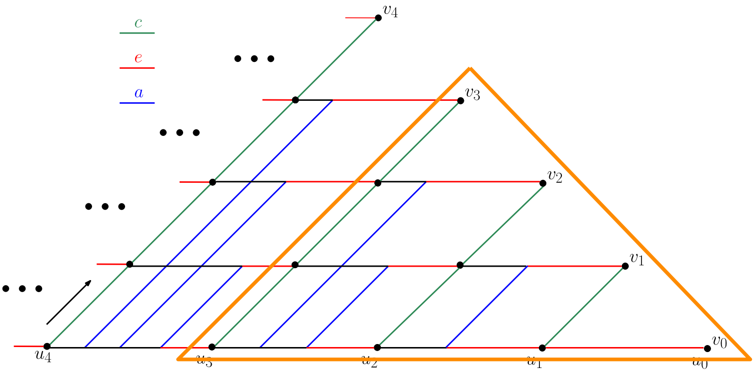
Paths from u_4 to v_k

$$P(u_4 \to v_k) = c$$



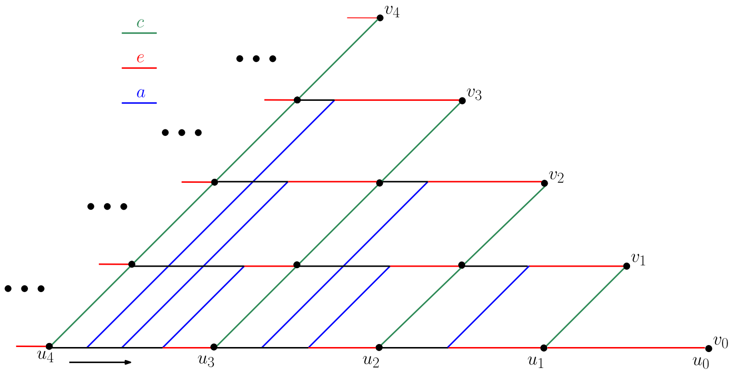
Paths from u_4 to v_k

$$P(u_4 \to v_k) = c$$



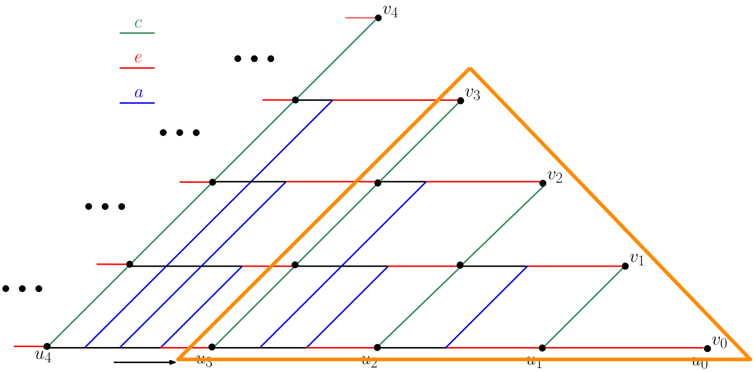
Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) +$$



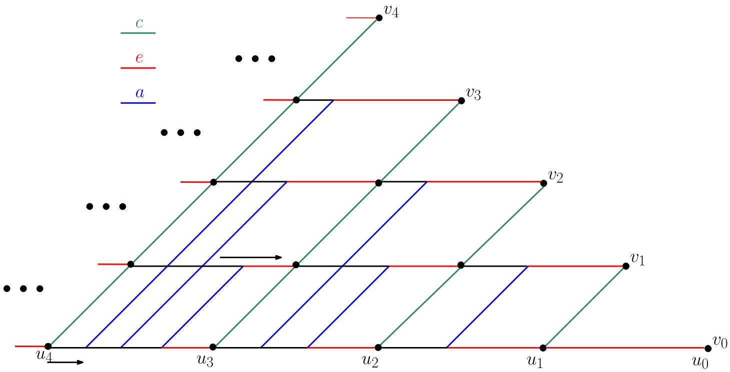
Paths from u_4 to v_k

$$P(u_4 \to v_k) = c P(u_3 \to v_{k-1}) + \binom{3}{0} e$$



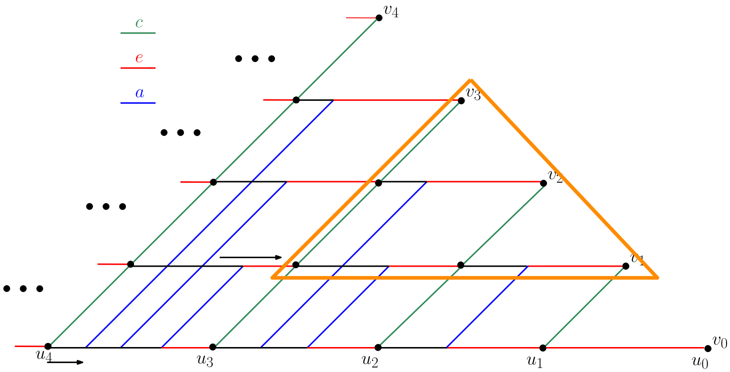
Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} eP(u_3 \to v_k) +$$



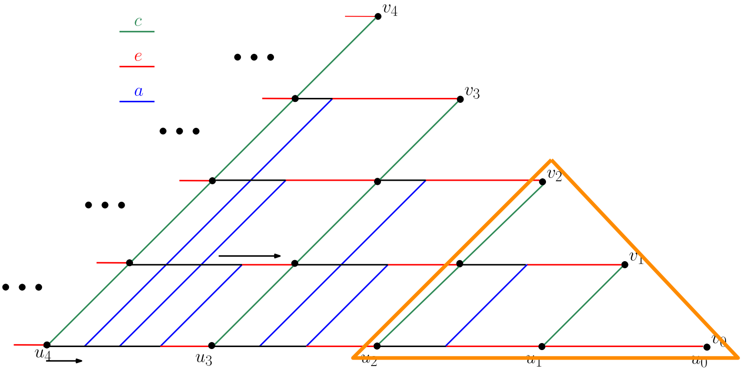
Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} eP(u_3 \to v_k) + \binom{3}{1} ae$$



Paths from u_4 to v_k

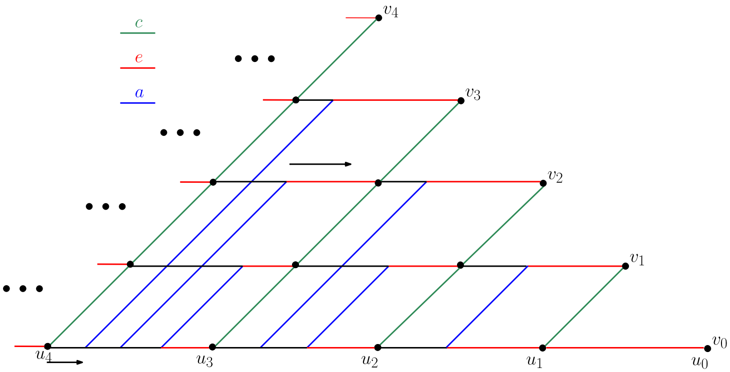
$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} eP(u_3 \to v_k) + \binom{3}{1} ae$$



Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} eP(u_3 \to v_k) +$$

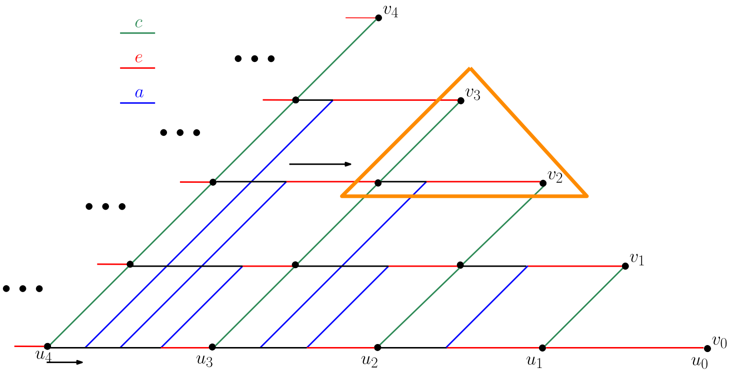
$$\binom{3}{1} aeP(u_2 \to v_{k-1}) +$$



Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} eP(u_3 \to v_k) +$$

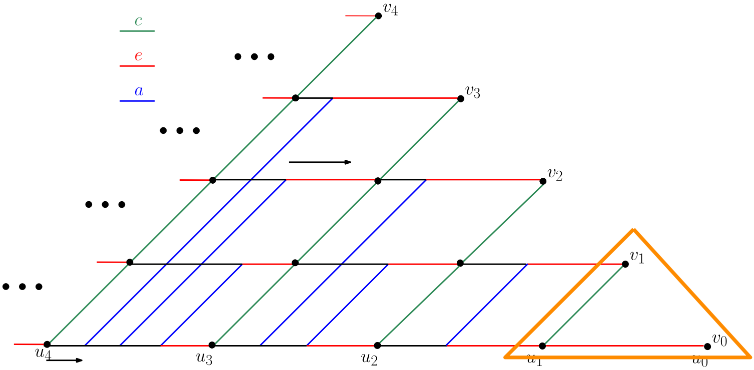
$$\binom{3}{1} aeP(u_2 \to v_{k-1}) +$$



Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} eP(u_3 \to v_k) +$$

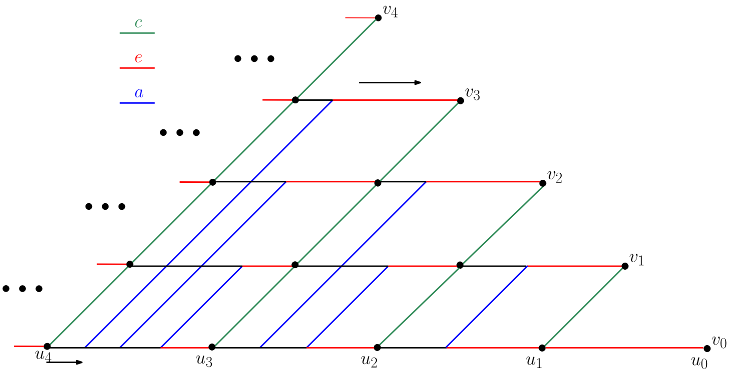
$$\binom{3}{1} aeP(u_2 \to v_{k-1}) + \binom{3}{2} a^2 e$$



Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} eP(u_3 \to v_k) +$$

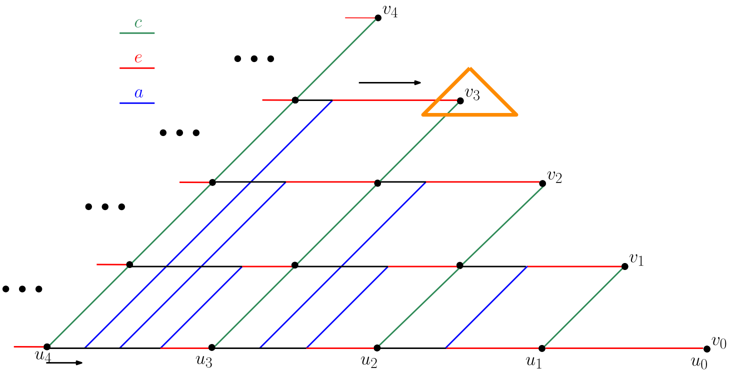
$$\binom{3}{1} aeP(u_2 \to v_{k-1}) + \binom{3}{2} a^2 eP(u_1 \to v_{k-2}) +$$



Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} eP(u_3 \to v_k) +$$

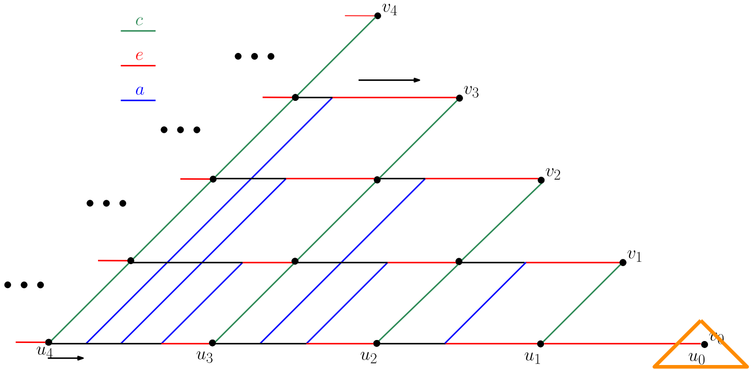
$$\binom{3}{1} aeP(u_2 \to v_{k-1}) + \binom{3}{2} a^2 eP(u_1 \to v_{k-2}) +$$



Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} eP(u_3 \to v_k) +$$

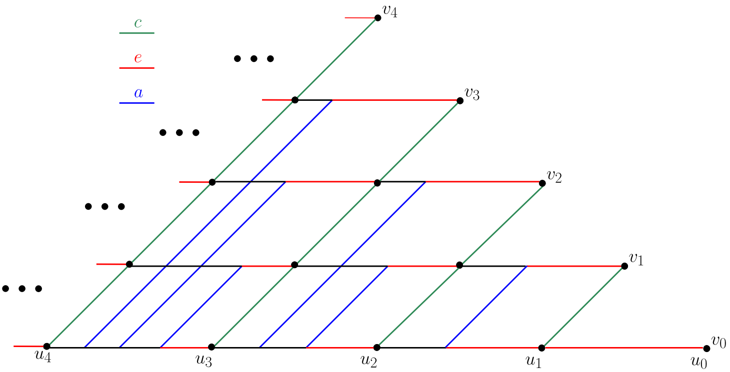
$$\binom{3}{1} aeP(u_2 \to v_{k-1}) + \binom{3}{2} a^2 eP(u_1 \to v_{k-2}) + \binom{3}{3} a^3 e$$



Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} eP(u_3 \to v_k) +$$

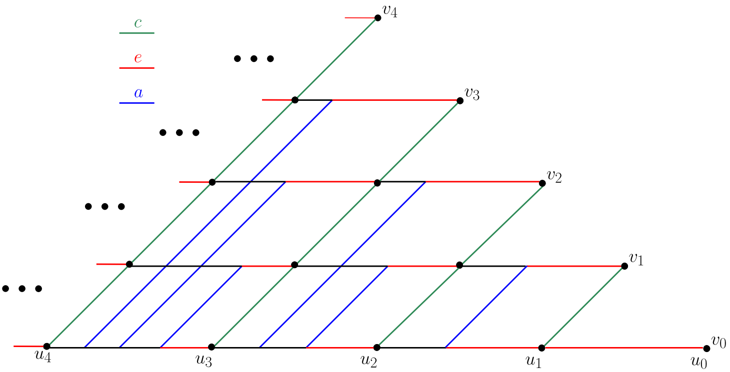
$$\binom{3}{1} aeP(u_2 \to v_{k-1}) + \binom{3}{2} a^2 eP(u_1 \to v_{k-2}) + \binom{3}{3} a^3 eP(u_0 \to v_{k-3})$$



Paths in network satisfy

$$P(u_n \to v_k) = cP(u_{n-1} \to v_{k-1}) + \sum_{m=0}^{n-1} \binom{n-1}{m} a^m e P(u_{n-1-m}, v_{k-m})$$

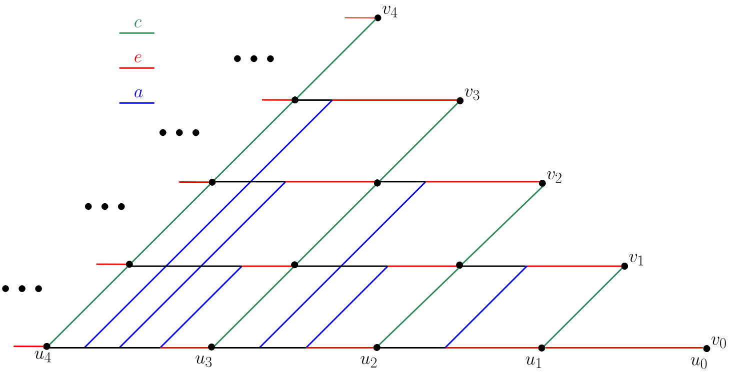
for $n \geq 1$, where $P(u_n \to v_k) = 0$ if $n < 0$ or $k < 0$.



The entries of $\mathbf{T}(a, c, e)$ satisfy the recurrence:

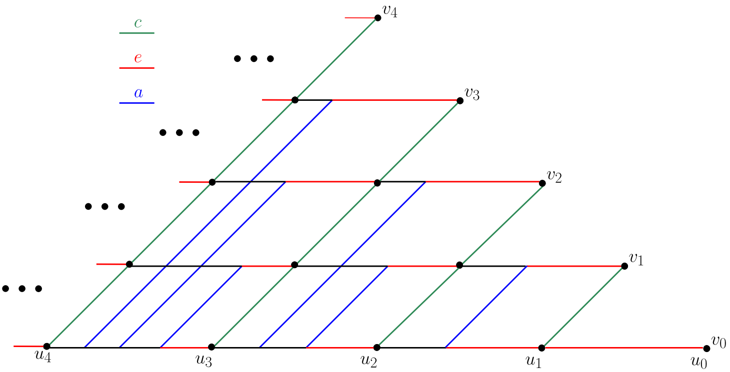
$$T(n,k) = cT(n-1,k-1) + \sum_{m=0}^{n-1} \binom{n-1}{m} a^m e T(n-1-m, k-m)$$

for $n \geq 1$, where $T(n,k) = 0$ if $n < 0$ or $k < 0$.



Theorem (X. Chen, B. Deb, A. Dyachenko, TG, A. D. Sokal '21)

The matrix $\mathbf{T}(a, c, e)$ *is coefficientwise totally positive in the indeterminates* a, c, e.



Conjecture (X. Chen, B. Deb, A. Dyachenko, TG, A. D. Sokal '21)

The matrix $\mathbf{T}(a, c, d, e, f, g)$ *is* **coefficientwise totally positive**.