Springer fibers and the Delta Conjecture at $t = 0$

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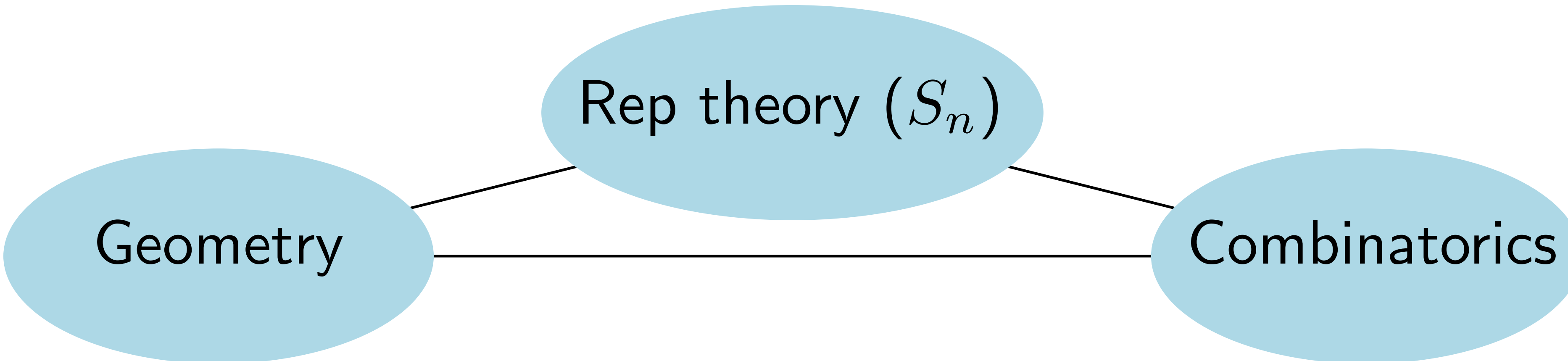
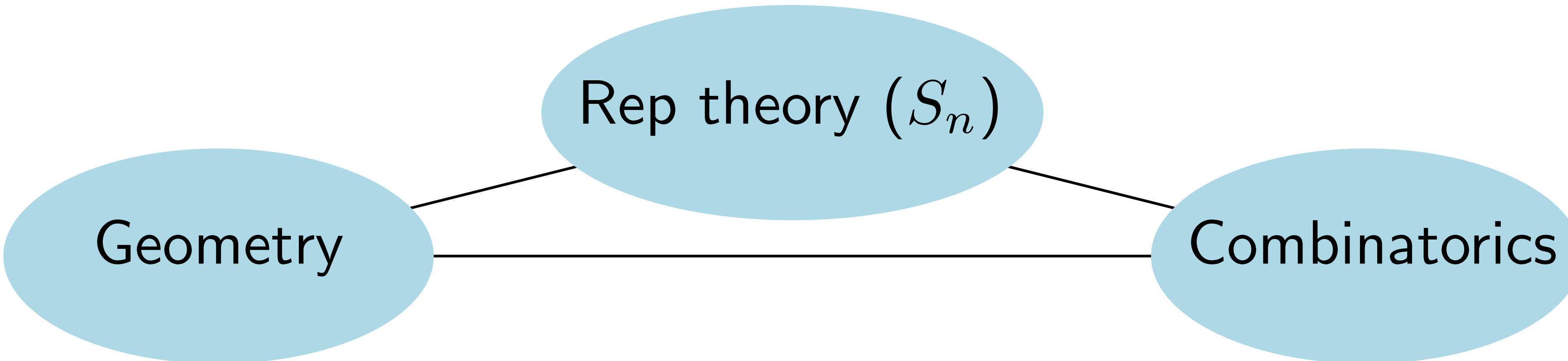
Joint work with Jake Levinson (Simon Fraser University) and Alex Woo (University of Idaho)

Geometric representations



Flag variety	Coinvariant ring	Permutations
$\mathrm{Fl}(\mathbb{C}^n) = G/B$	$R_n \cong H^*(\mathcal{B})$	S_n

Geometric representations



Flag variety
$\mathrm{Fl}(\mathbb{C}^n) = G/B$

Springer fiber
$\mathcal{B}^\lambda$

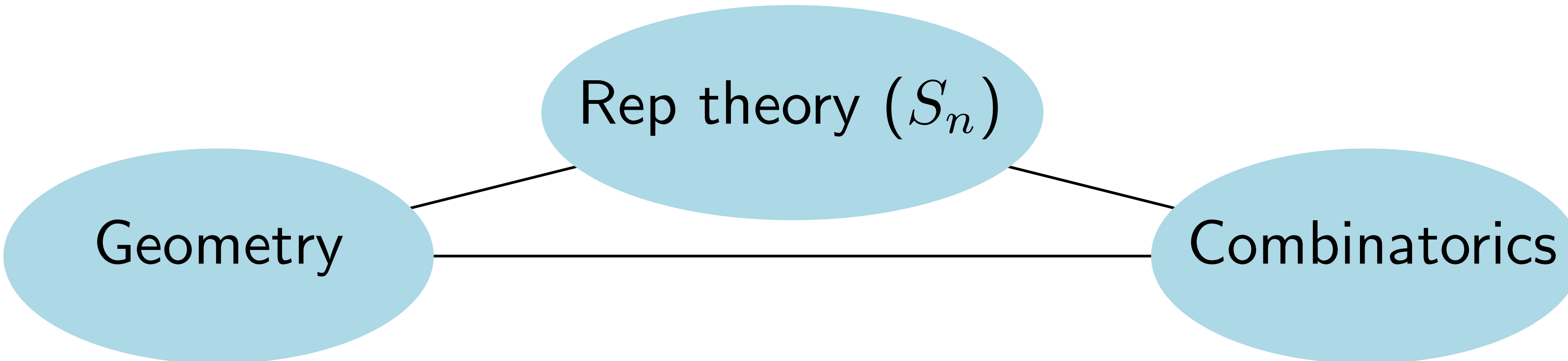
Coinvariant ring
$R_n \cong H^*(\mathcal{B})$

Garsia-Procesi module
$R_\lambda \cong H^*(\mathcal{B}^\lambda)$

Permutations
S_n

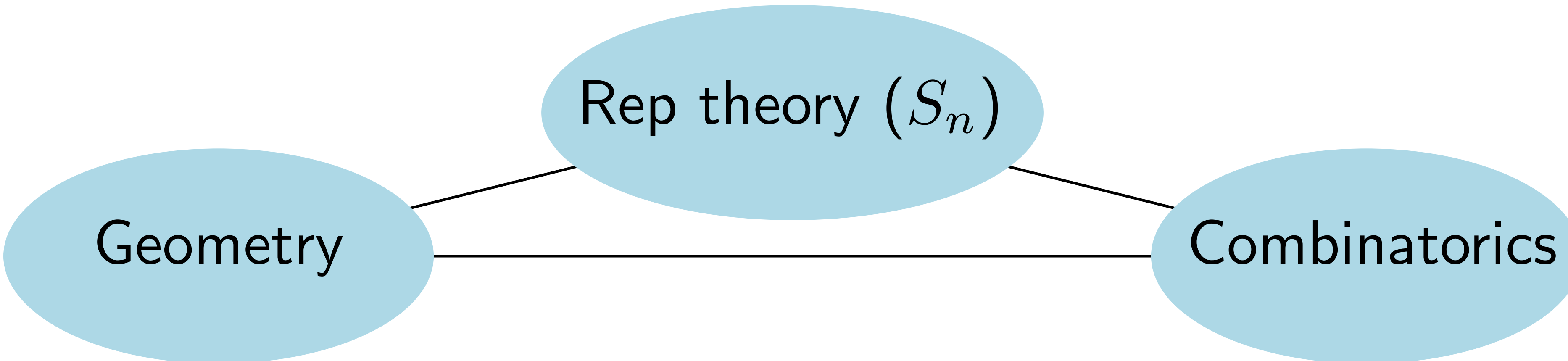
Tabloids of
shape λ

Geometric representations



Flag variety $\mathrm{Fl}(\mathbb{C}^n) = G/B$	Coinvariant ring $R_n \cong H^*(\mathcal{B})$	Permutations S_n
Springer fiber $\mathcal{B}^\lambda$	Garsia-Procesi module $R_\lambda \cong H^*(\mathcal{B}^\lambda)$	Tabloids of shape λ
Span. line config. (PR) $X_{n,k}$	Gen. coinv. ring (HRS) $R_{n,k} \cong H^*(X_{n,k})$	Ordered set partitions k blocks

Geometric representations



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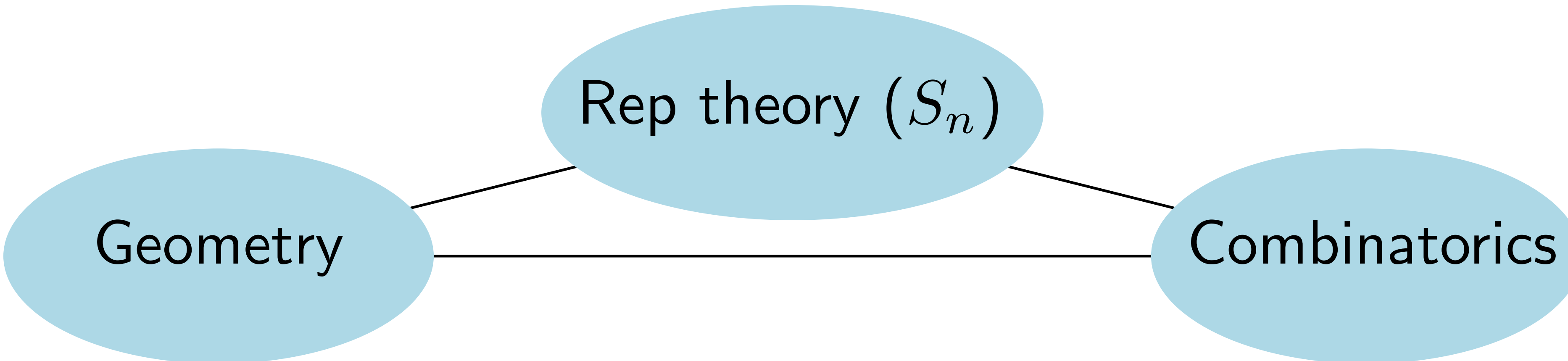

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Δ-Springer variety (GLW) $Y_{n,\lambda}$	$R_{n,\lambda}$ rings (G.) $R_{n,\lambda} = H^*(Y_{n,\lambda})$	(n,λ)-OSPs or partial row decr. fillings

Background: Springer fibers

Complete flag variety $\mathrm{Fl}(\mathbb{C}^n)$

The *complete flag variety* is:

$$\mathrm{Fl}(\mathbb{C}^n) = \{V_\bullet = (V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \mid \dim(V_i) = i\}$$

- $\mathrm{Fl}(\mathbb{C}^n)$ is partitioned into *Schubert cells* indexed by permutations $w \in S_n$:

  $$C_w = \{V_\bullet \mid \dim(V_i \cap E_j) = \#\{p \leq i \,:\, w(p) \leq j\}\},$$

  where $E_j = \mathrm{span}\{e_1, e_2, \ldots, e_j\}$.

- The Poincaré duals of the homology classes $[\overline{C}_w]$ are a basis of $H^*(\mathrm{Fl}(\mathbb{C}^n))$ (represented by Schubert polynomials), so $H^*(\mathrm{Fl}(\mathbb{C}^n))$ is free of rank $n!$.

- $H^*(\mathrm{Fl}(\mathbb{C}^n)) \cong \mathbb{Z}[x_1, \ldots, x_n]/(\mathbb{Z}[x_1, \ldots, x_n])_+^{S_n}$

Springer fiber $\mathcal{B}_\lambda$

Let N_λ = a nilpotent matrix with Jordan type $\lambda \vdash n$.
The *Springer fiber* is:

$$\mathcal{B}^\lambda := \{V_\bullet \in \mathrm{Fl}(\mathbb{C}^n) \,:\, N_\lambda V_i \subseteq V_i \text{ for all } i\}.$$

Examples:

- $\lambda = (1^n)$, $N_{(1^n)} = 0$, $\mathcal{B}^{(1^n)} = \mathrm{Fl}(\mathbb{C}^n)$.
- $\lambda = (n)$,

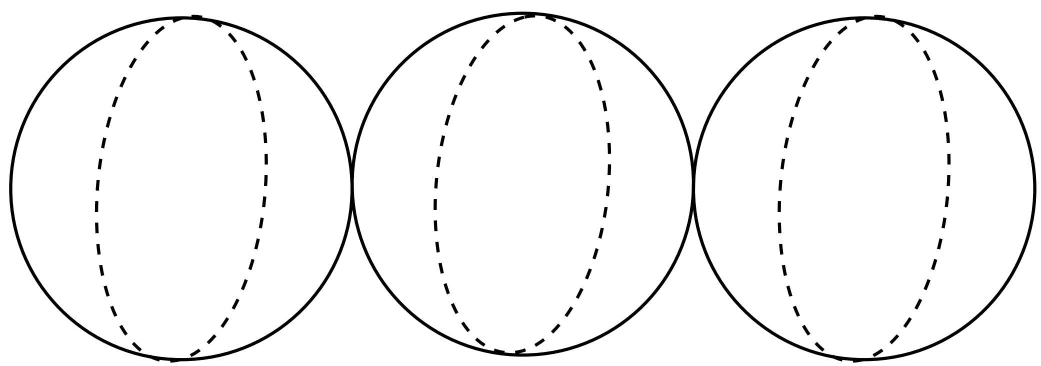
$$N_{(n)} = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & \\ \vdots & & & \ddots \end{bmatrix} \qquad \mathcal{B}^{(n)} = \{E_\bullet\}.$$

(a single point)

Combinatorial properties of $\mathcal{B}^\lambda$

An *affine paving* is a type of cell decomposition.

- $\mathcal{B}^\lambda$ has an *affine paving* with cells $C_w \cap \mathcal{B}^\lambda$.
- Nonempty cells $C_w \cap \mathcal{B}^\lambda \leftrightarrow$ row-decreasing fillings of λ
- Irreducible components of $\mathcal{B}^\lambda \leftrightarrow \mathrm{SYT}(\lambda)$
- Each component has dim $n(\lambda) = \sum_i \binom{\lambda_i'}{2}$

$\mathcal{B}^{(3,1)} \cong$ 

3	2	1
4		

4	2	1
3		

4	3	1
2		

4	3	2
1		

Springer correspondence

Let S^λ be the *Specht module*, the irreducible S_n-module indexed by λ.

Theorem (Springer '76)

There is an action of S_n on $H^*(\mathcal{B}^\lambda)$, and

$$H^{top}(\mathcal{B}^\lambda; \mathbb{Q}) \cong S^\lambda.$$

Springer fibers *geometrically construct* the Specht modules.

Theorem (Hotta–Springer '77, Garsia–Procesi '92)

$$\mathrm{Frob}(H^*(\mathcal{B}^\lambda; \mathbb{Q}); q) = \widetilde{H}_\lambda(\mathbf{x}; q)$$

Background: The Delta Conjecture

Delta Conjecture

- Haiman proved the graded Frobenius characteristic of the *diagonal coinvariant ring* $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]/(\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n])_+^{S_n}$ is
$$\Delta'_{e_{n-1}} e_n(q,t).$$
- Δ'_f is the Macdonald eigenoperator defined by Bergeron, Garsia, Haiman, and Tesler.
- The Shuffle Theorem, proved by Carlsson and Mellit, gives a combinatorial formula for $\Delta'_{e_{n-1}} e_n(q,t)$.
- For $k \leq n$, the Delta Conjecture of Haglund, Remmel, and Wilson gives combinatorial formulas for the symmetric function $\Delta'_{e_{k-1}} e_n(q,t)$.
- The "rise half" of the conjecture was proved by D'Adderio–Mellit and Blasiak–Haiman–Morse–Pun-Seelinger!

$R_{n,k}$ and $X_{n,k}$

Haglund–Rhoades–Shimozono found an algebraic realization of the $t = 0$ case of the Delta Conjecture:

$$R_{n,k} := \mathbb{Q}[\mathbf{x}_n]/\langle x_1^k, \ldots, x_n^k, e_n, e_{n-1}, \ldots, e_{n-k+1} \rangle$$

Theorem (Haglund–Rhoades–Shimozono 2018)

$$\mathrm{Frob}(R_{n,k}; q) = (\omega \circ \mathrm{rev}_q)\, \Delta'_{e_{k-1}} e_n(q, 0)$$

Pawlowski and Rhoades found a geometric realization:

Theorem (Pawlowski–Rhoades 2019)

Let $X_{n,k}$ be the space of n-tuples of lines in $\mathbb{C}^k$ that span $\mathbb{C}^k$,

- $H^*(X_{n,k}) \cong R_{n,k}$ *as graded rings.*
- $X_{n,k}$ *has a paving with cells indexed by OSPs of $[n]$ into k blocks.*

The Δ-Springer varieties

Spaltenstein variety

Let α be a (strong) composition of size $\leq n$.

$\mathrm{Fl}_\alpha(\mathbb{C}^n)$ is the *partial flag variety* of $V_\bullet = (V_1, V_2, \ldots, V_{\ell(\alpha)})$ where $\dim(V_i/V_{i-1}) = \alpha_i$.

The *Spaltenstein variety* is

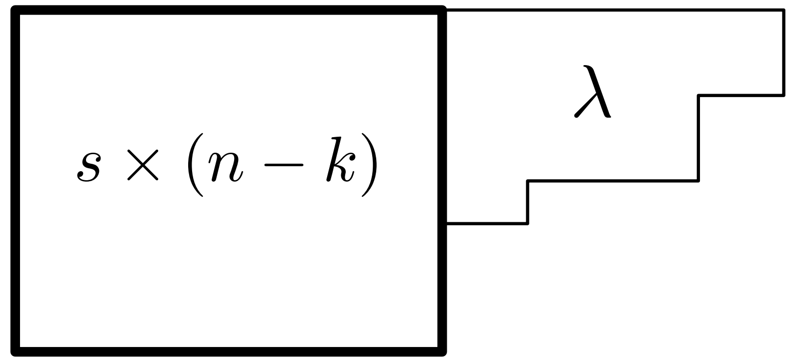
$$\mathcal{B}^\lambda_{\alpha} := \{V_\bullet \in \mathrm{Fl}_{\alpha}(\mathbb{C}^n) : N_\lambda V_i \subseteq V_{i-1} \text{ for all } i \leq \ell(\alpha)\}.$$

(Fresse, Brundan–Ostrik) $\mathcal{B}^\lambda_\alpha$ has an affine paving

Different from Steinberg variety, which also has an affine paving (Steinberg, Shimomura, Precup–Tymoczko)

Definition of the Δ-Springer variety

Let $k \leq n$, let $\lambda \vdash k$, and $s \geq \ell(\lambda)$.

$\Lambda :=$ 

Let $\pi_n : \mathrm{Fl}_{(1^n,(s-1)^{n-k})}(\mathbb{C}^{|\Lambda|}) \to \mathrm{Fl}_{(1^n)}(\mathbb{C}^{|\Lambda|})$ projection.

Definition 1 (G.–Levinson–Woo 2021)

$$Y_{n,\lambda,s} := \pi_n\left(\mathcal{B}^{\Lambda}_{(1^n,(s-1)^{n-k})}\right).$$

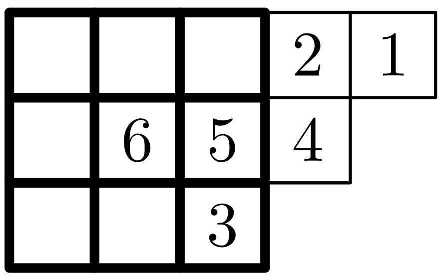
Definition 2 (G.–Levinson–Woo 2021)

$$Y_{n,\lambda,s} := \{V_\bullet \in \mathrm{Fl}_{(1^n)}(\mathbb{C}^{|\Lambda|}) \mid N_\Lambda V_i \subseteq V_i,\ \mathrm{im}(N_\Lambda^{n-k}) \subseteq V_n\}$$

Paving of the Δ-Springer variety

Theorem (G.–Levinson–Woo 2021)

- *$Y_{n,\lambda,s}$ has an affine paving with cells $C_w \cap Y_{n,\lambda,s}$.*
- *Cells $\leftrightarrow$ Row-decreasing right-justified partial fillings of Λ with $1,\ldots,n$ s.t. λ is completely filled.*



Cell in $Y_{6,(2,1),3}$

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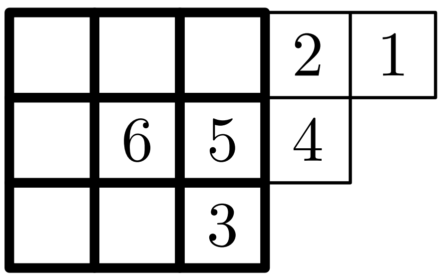
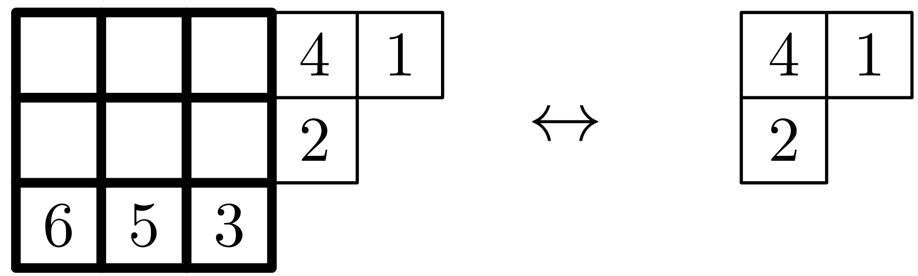


Cell in $Y_{6,(2,1),3}$



Max dim cell

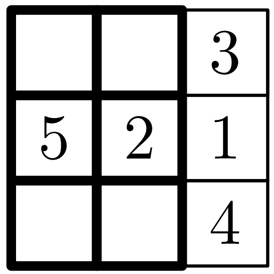
Theorem (G.–Levinson–Woo 2021)

- *When $s > \ell(\lambda)$,*
  Irred. comp.$\leftrightarrow$ row/column decreasing injective tableaux on λ
- *Each component has dimension $n(\lambda) + (n-k)(s-1)$.*

Connection to Delta Conjecture

Example: $\lambda = (1^k)$, $s = k$

In this case, the cells are in bijection with ordered set partitions.



$\leftrightarrow \quad (\{3\}, \{1,2,5\}, \{4\})$

Cell in $Y_{5,(1,1,1),3}$

Theorem (G.–Levinson–Woo 2021)

When $\lambda = (1^k)$ *and* $s = k$,

$$H^*(Y_{n,(1^k),k}) \cong R_{n,k}.$$

So $Y_{n,\lambda,s}$ gives a *compact* (but singular) geometric realization of the Delta Conjecture at $t = 0$.

Generalization of Springer correspondence

More generally, $H^*(Y_{n,\lambda,s})$ has a quotient ring presentation $R_{n,\lambda,s}$ that matches the ring from my thesis,

$$H^*(Y_{n,\lambda,s}) \cong R_{n,\lambda,s} = \mathbb{Q}[\mathbf{x}_n]/I_{n,\lambda,s}.$$

There is an S_n-action on $H^*(Y_{n,\lambda,s})$.

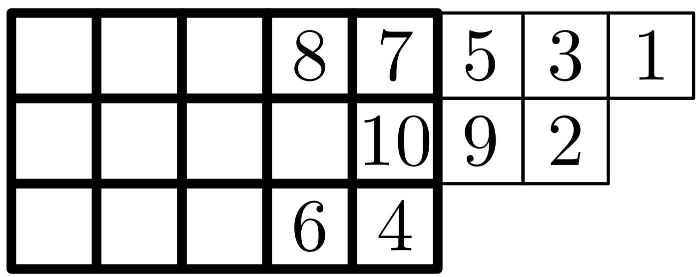
Theorem (G. 2020 + G.–Levinson–Woo 2021)

When $s > \ell(\lambda)$,

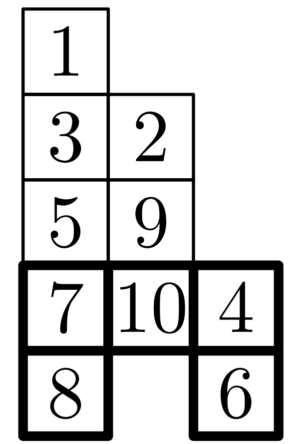
$$H^{top}(Y_{n,\lambda,s};\mathbb{Q}) \cong \mathrm{Ind}\uparrow_{S_k \times S_{n-k}}^{S_n} (S^\lambda \otimes \mathbf{1}).$$

So $Y_{n,\lambda,s}$ *geometrically constructs* induction products of the form $S^\lambda \circ \mathbf{1}$ (given by Pieri rule).

Frobenius characteristic formula



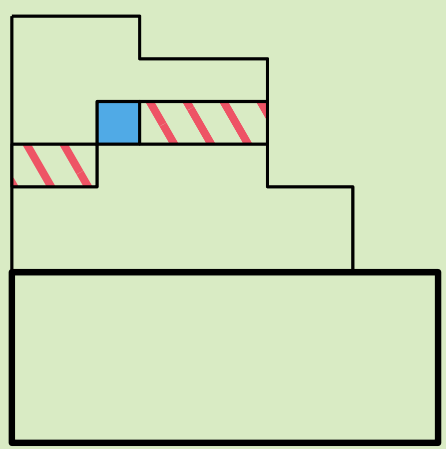
$\circlearrowleft$ $\varphi =$



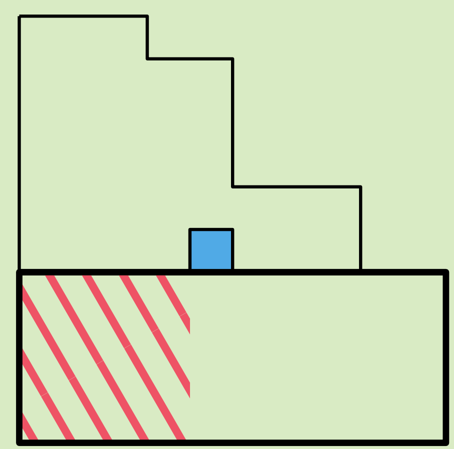
basement cells

Definition (Rhoades–Yu–Zhao 2020)

An *attacking pair* is a pair of labels $b > a$ of the form:



or



Let $\mathrm{inv}(\varphi) = \#$ attacking pairs

$+ \sum_i (i-1)\,\#($basement cells in col. $i)$.

Frobenius characteristic formula

Rhoades–Yu–Zhao used inv to find the Hilbert series and space of *harmonics* for $R_{n,\lambda,s}$.

Let $\mathrm{ECI}_{n,\lambda,s}$ be the fillings obtained by allowing repeated labels in a column.

Theorem (G. 2020 + G.–Levinson–Woo 2021)

$$\mathrm{Frob}(H^*(Y_{n,\lambda,s};\mathbb{Q});q) = \sum_{\varphi \in \mathrm{ECI}_{n,\lambda,s}} q^{\mathrm{inv}(\varphi)} \mathbf{x}^{\varphi}.$$

There is also a *diagonal inversion* statistic $\mathrm{dinv}(\varphi)$ that gives an LLT expansion.

Further properties

- (G.-Levinson-Woo 2021) The cohomology of $\varinjlim_s Y_{n,\lambda,s}$ is the coordinate ring of the scheme of diagonal "rank deficient" matrices.

Forthcoming:

- (G., 2022) $\mathrm{inv}(\varphi)$ and $\mathrm{dinv}(\varphi)$ compute the dimensions of cells (under different ordered bases).

- (G., 2022) Counting points of $Y_{n,\lambda,s}$ over $\mathbb{F}_q$ can be used to find a $\widetilde{H}_\mu(\mathbf{x};q)$ expansion.

Further Directions

- Geometric properties of $Y_{n,\lambda,s}$? Singularities of cmpts?
- Connections to Δ' operators?
- q-t parameter version of $\mathrm{Frob}(H^*(Y_{n,\lambda,s}); q)$?

$$\widetilde{H}_\lambda(\mathbf{x}; q, t) \xrightarrow{t=0} \mathrm{Frob}(H^*(\mathcal{B}^\lambda; \mathbb{Q}); q)$$

$$\Delta'_{e_{k-1}} e_n(q, t) \xrightarrow{\text{``}t=0\text{''}} \mathrm{Frob}(H^*(Y_{n,(1^k),k}; \mathbb{Q}); q)$$

$$?? \xrightarrow{t=0} \mathrm{Frob}(H^*(Y_{n,\lambda,s}; \mathbb{Q}); q)$$

Thanks for listening!

The rings $R_{n,\lambda}$

Let $k \leq n$ **and** $\lambda \vdash k$**.**

Write $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n \geq 0)$

Let $p^n_m(\lambda) = \lambda'_n + \cdots + \lambda'_{n-m+1}$.

$$I_{n,\lambda} \coloneqq \langle e_d(S) \,:\, S \subseteq \mathbf{x}_n,\, d > |S| - p^n_{|S|}(\lambda) \rangle,$$

$$R_{n,\lambda} \coloneqq \mathbb{Q}[\mathbf{x}_n]/I_{n,\lambda}$$

The rings $R_{n,\lambda}$

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Write $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n \geq 0)$

Let $p^n_m(\lambda) = \lambda'_n + \cdots + \lambda'_{n-m+1}$.

$$I_{n,\lambda} := \langle e_d(S) \,:\, S \subseteq \mathbf{x}_n,\, d > |S| - p^n_{|S|}(\lambda) \rangle,$$
$$R_{n,\lambda} := \mathbb{Q}[\mathbf{x}_n]/I_{n,\lambda}$$

For $s \geq \ell(\lambda)$, also define

$$I_{n,\lambda,s} := I_{n,\lambda} + \langle x_1^s, \ldots, x_n^s \rangle,$$
$$R_{n,\lambda,s} := \mathbb{Q}[\mathbf{x}_n]/I_{n,\lambda,s}$$

Hall–Littlewood expansion

- q-binomial coefficient:

$$\begin{bmatrix} a \\ b \end{bmatrix}_q := \frac{[a]_q!}{[b]_q![a-b]_q!}$$

- $n(\mu, \lambda) = \sum_{i \geq 1} \binom{\mu'_i - \lambda'_i}{2}$

Theorem (G., 2020)

$\mathrm{Frob}(H^*(Y_{n,\lambda,s}); q)$ has the following expansion

$$\mathrm{rev}_q \left[\sum_{\substack{\mu \vdash n \\ \ell(\mu)=\ell(\lambda) \\ \mu_i \geq \lambda_i \forall i}} q^{n(\mu,\lambda)} \prod_{i \geq 0} \begin{bmatrix} \mu'_i - \lambda'_{i+1} \\ \mu'_i - \mu'_{i+1} \end{bmatrix}_q H_\mu(\mathbf{x}; q) \right].$$

where $\mu'_0 := s$.

Rank varieties

(Eisenbud–Saltman) Given $k \leq n$ and $\lambda \vdash k$, let

$$\overline{O}_{n,\lambda} = \{X \in \mathfrak{gl}_n \ : \ \mathrm{rk}(X^i) \leq (n-k) + p_{n-i}(\lambda) \text{ for all } i\}.$$

Eisenbud-Saltman conjectured an explicit generating set for $I(\overline{O}_{n,\lambda})$, which Weyman (1989) proved.

Corollary (G.–Levinson–Woo 2021)

As graded rings and graded S_n-modules,

$$H^*\left(\varinjlim_s Y_{n,\lambda,s}; \mathbb{Q}\right) \cong \mathbb{Q}[\overline{O}_{n,\lambda'} \cap \mathfrak{t}].$$