Weighted posets and the enriched monomial basis of QSym [slides]

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these slides: https://github.com/darijgr/fpsac2022/raw/main/fpsaclive/talk.pdf
extended abstract: https://www.mat.univie.ac.at/~slc/wpapers/FPSAC2022/58Grinberg.pdf

Summary of our work

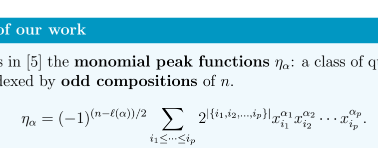
Hsiao defines in [5] the **monomial peak functions** η_α, a class of quasisymmetric functions indexed by **odd compositions** of n.

They provide a monomial-like basis to Stembridge's algebra of peaks [6] and are related to Stembridge peak functions K_α through

$$K_\alpha = \sum_{\beta \in \mathrm{OddComp}(n);\ \mathrm{Peak}(\beta)\subseteq \mathrm{Peak}(\alpha)} \eta_\beta .$$

In the present work:

- We show that monomial peak functions may be extended to a **basis of (the whole) QSym**. We name this new basis the **enriched monomial basis of QSym**.
- We relate it to other bases of QSym, compute its **antipode** and **coproduct**.
- We introduce **weighted posets** and their **enriched P-partitions** (generalising both the weighted posets of [2] and the enriched P-partitions of [6]), whose generating functions give a **universal framework** for many types of quasisymmetric functions.



- We use our framework to compute the **product** of two enriched monomials:

$$\eta_\alpha \eta_\beta = \sum (-1)^{|I|} \eta_{\gamma\downarrow\downarrow I}.$$

1. Notation and basic definitions

1.1. Compositions and permutations statistics

- $\mathbb{P} = \{1,2,3,\ldots\}$, $\mathbb{N} = \{0,1,\ldots\}$, $[n] = \{1,2,\ldots,n\}$, S_n the symmetric group on $[n]$.
- $\mathrm{Comp}(n)$ is the set of compositions $\alpha = (\alpha_1,\ldots,\alpha_p)$ of n, and $\mathrm{OddComp}(n)$ is the set of *odd compositions* of n (containing only odd integers). We let $\ell(\alpha) = p$.
- The *descent set* $\mathrm{Des}(\pi)$ of $\pi \in S_n$ is $\mathrm{Des}(\pi) = \{1 \le i \le n-1 \mid \pi(i) > \pi(i+1)\}$.
- The *peak set* $\mathrm{Peak}(\pi)$ of π is $\mathrm{Peak}(\pi) = \{2 \le i \le n-1 \mid \pi(i-1) < \pi(i) > \pi(i+1)\}$.
- Compositions of n are in bijection with descent sets of permutations in S_n, while odd compositions of n are in bijection with peak sets. Denote $\mathrm{Des}(\alpha)$ (resp. $\mathrm{Peak}(\alpha)$) the corresponding set.

1.2. Quasisymmetric functions

- For any $\alpha \in \mathrm{Comp}(n)$ with $\alpha = (\alpha_1,\ldots,\alpha_p)$, define the *monomial quasisymmetric function* M_α and the *fundamental quasisymmetric function* L_α by

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_p} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_p}^{\alpha_p}, \qquad L_\alpha = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n;\\ i_j < i_{j+1} \text{ if } j \in \mathrm{Des}(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

- As an example, for $n = 3$, we have

$$M_{(2,1)} = \sum_{i<j} x_i^2 x_j = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + \cdots,$$

$$L_{(2,1)} = \sum_{i \le j < k} x_i x_j x_k = x_1^2x_2 + x_1^2x_3 + x_1x_2x_3 + \cdots.$$

- The families $(M_\alpha)_{n\in\mathbb{N},\ \alpha\in\mathrm{Comp}(n)}$ and $(L_\alpha)_{n\in\mathbb{N},\ \alpha\in\mathrm{Comp}(n)}$ are two bases of the $\mathbf{k}$-module QSym. They are related through

$$L_\alpha = \sum_{\substack{\beta \in \mathrm{Comp}(n);\\ \mathrm{Des}(\beta) \supseteq \mathrm{Des}(\alpha)}} M_\beta. \tag{1}$$

1.3. Peak and monomial peak functions

- In [6], Stembridge introduces *peak quasisymmetric functions*: Given $n \in \mathbb{N}$ and $\alpha \in \mathrm{OddComp}(n)$, the corresponding function is

$$K_\alpha = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n;\\ j \in \mathrm{Peak}(\alpha) \Longrightarrow \neg(i_{j-1} = i_j = i_{j+1})}} 2^{|\{i_1,i_2,\ldots,i_n\}|} x_{i_1} x_{i_2} \cdots x_{i_n} \in \mathrm{QSym}. \tag{2}$$

- Hsiao defines in [5] the *monomial peak functions*: For any odd composition $\alpha = (\alpha_1,\ldots,\alpha_p)$ of n,

$$\eta_\alpha = (-1)^{(n-\ell(\alpha))/2} \sum_{i_1 < i_2 < \cdots < i_p} 2^{\ell(\alpha)} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_p}^{\alpha_p}. \tag{3}$$

- An identity similar to Equation (1) relates peak and monomial peak functions:

$$K_\alpha = \sum_{\substack{\beta \in \mathrm{OddComp}(n);\\ \mathrm{Peak}(\beta) \subseteq \mathrm{Peak}(\alpha)}} \eta_\beta.$$

1.4. Posets and P-partitions

- A *labelled poset* $P = ([n], <_P)$ is an arbitrary partial order $<_P$ on the set $[n]$.
- Let $P = ([n], <_P)$ be a labelled poset. A *P-partition* is a map $f : [n] \to \mathbb{P}$ that satisfies the following two conditions:
  (i) If $i <_P j$, then $f(i) \le f(j)$.
  (ii) If $i <_P j$ and $i > j$, then $f(i) < f(j)$.
- Let $\mathbb{P}^\pm$ be the (manually) totally ordered set

$$\mathbb{P}^\pm = \{-1 < 1 < -2 < 2 < -3 < 3 < \cdots\} = \mathbb{P} \cup (-\mathbb{P}).$$

- Let $P = ([n], <_P)$ be a labelled poset. An *enriched P-partition* is a map $f : [n] \to \mathbb{P}^\pm$ that satisfies the following two conditions:
  (i) If $i <_P j$ and $i < j$, then $f(i) < f(j)$ or $f(i) = f(j) \in \mathbb{P}$.
  (ii) If $i <_P j$ and $i > j$, then $f(i) < f(j)$ or $f(i) = f(j) \in -\mathbb{P}$.
- A more general concept was defined in [3]:
  Let $\mathcal{Z}$ be a subset of $\mathbb{P}^\pm$, and $P = ([n], <_P)$ a labelled poset. A *$\mathcal{Z}$-enriched P-partition* is an enriched P-partition f such that $f(i) \in \mathcal{Z}$ for all i. Let $\mathcal{A}_\mathcal{Z}(P)$ denote the set of $\mathcal{Z}$-enriched P-partitions.
- Let $X = \{x_1, x_2, x_3, \ldots\}$, $P = ([n], <_P)$ and $\mathcal{Z} \subseteq \mathbb{P}^\pm$. Define the *$\mathcal{Z}$-generating function of P* as the formal power series

$$\Gamma_\mathcal{Z}([n], <_P) = \sum_{f \in \mathcal{A}_\mathcal{Z}([n],<_P)} \prod_{1 \le i \le n} x_{|f(i)|}. \tag{4}$$

- Given $\pi \in S_n$, let $P_\pi = ([n], <_\pi)$ denote the labelled poset where the order relation $<_\pi$ is such that $\pi_1 <_\pi \pi_2 <_\pi \cdots <_\pi \pi_n$.

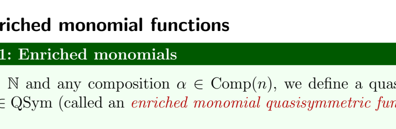


Figure 1: The labelled poset associated to permutation π.

For $\mathcal{Z} = \mathbb{P}$, this recovers Gessel's P-partition enumerator (called $\Gamma(P)$ in [1]).

- Set $L_\pi = \Gamma_\mathbb{P}([n], <_\pi)$ and $K_\pi = \Gamma_{\mathbb{P}^\pm}([n], <_\pi)$. The function L_π is equal to the fundamental quasisymmetric function L_α indexed by the unique composition α such that $\mathrm{Des}(\alpha) = \mathrm{Des}(\pi)$. Similarly, K_π is equal to the peak function K_α indexed by the unique odd composition α such that $\mathrm{Peak}(\alpha) = \mathrm{Peak}(\pi)$.
- Given two permutations $\pi \in S_n$ and $\sigma \in S_m$, we have

$$\Gamma_\mathcal{Z}([n], <_\pi) \cdot \Gamma_\mathcal{Z}([m], <_\sigma) = \sum_{\gamma \in \pi ⧢ \sigma} \Gamma_\mathcal{Z}([n+m], <_\gamma). \tag{5}$$

This recovers the known facts that

$$L_\pi L_\sigma = \sum_{\gamma \in \pi ⧢ \sigma} L_\gamma, \qquad K_\pi K_\sigma = \sum_{\gamma \in \pi ⧢ \sigma} K_\gamma.$$

2. The enriched monomial basis of QSym

2.1. The enriched monomial functions

Definition 1: Enriched monomials

For any $n \in \mathbb{N}$ and any composition $\alpha \in \mathrm{Comp}(n)$, we define a quasisymmetric function $\eta_\alpha \in \mathrm{QSym}$ (called an *enriched monomial quasisymmetric function*) by

$$\eta_\alpha = \sum_{\substack{\beta \in \mathrm{Comp}(n);\\ \mathrm{Des}(\beta) \subseteq \mathrm{Des}(\alpha)}} 2^{\ell(\beta)} M_\beta. \tag{6}$$

Examples

- Setting $n = 5$ and $\alpha = (1,3,1)$ in this definition, we obtain

$$\begin{aligned}\eta_{(1,3,1)} &= \sum_{\substack{\beta \in \mathrm{Comp}(5);\\ \mathrm{Des}(\beta) \subseteq \{1,4\}}} 2^{\ell(\beta)} M_\beta \qquad (\text{since } \mathrm{Des}(1,3,1) = \{1,4\})\\ &= 2^{\ell(5)} M_{(5)} + 2^{\ell(1,4)} M_{(1,4)} + 2^{\ell(4,1)} M_{(4,1)} + 2^{\ell(1,3,1)} M_{(1,3,1)}\\ &= 2M_{(5)} + 4M_{(1,4)} + 4M_{(4,1)} + 8M_{(1,3,1)}.\end{aligned}$$

- For any positive integer n, we have $\eta_{(n)} = 2M_{(n)}$ (since the composition (n) satisfies $\mathrm{Des}(n) = \varnothing$).
- When α is odd, this recover's Hsiao's monomial peak η_α up to sign.

Proposition 1: Power series expansion

Let $n \in \mathbb{N}$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p) \in \mathrm{Comp}(n)$. Then,

$$\begin{aligned}\eta_\alpha &= \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n;\\ j \notin \mathrm{Des}(\alpha) \Longrightarrow i_j = i_{j+1}}} 2^{|\{i_1,i_2,\ldots,i_n\}|} x_{i_1} x_{i_2} \cdots x_{i_n}\\ &= \sum_{i_1 \le i_2 \le \cdots \le i_p} 2^{|\{i_1,i_2,\ldots,i_p\}|} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_p}^{\alpha_p}.\end{aligned}$$

Proposition 2: Relation to fundamental basis

Let n be a positive integer. Let $\alpha \in \mathrm{Comp}(n)$. Then,

$$\eta_\alpha = 2 \sum_{\gamma \in \mathrm{Comp}(n)} (-1)^{|\mathrm{Des}(\gamma) \setminus \mathrm{Des}(\alpha)|} L_\gamma$$

and

$$2^n L_\alpha = \sum_{\gamma \in \mathrm{Comp}(n)} (-1)^{|\mathrm{Des}(\alpha) \setminus \mathrm{Des}(\gamma)|} \eta_\gamma.$$

Proposition 3: Expansion of monomial functions in enriched monomial functions

For any $n \in \mathbb{N}$ and $\beta \in \mathrm{Comp}(n)$, we have

$$2^{\ell(\beta)} M_\beta = \sum_{\substack{\alpha \in \mathrm{Comp}(n);\\ \mathrm{Des}(\alpha) \subseteq \mathrm{Des}(\beta)}} (-1)^{\ell(\beta) - \ell(\alpha)} \eta_\alpha.$$

2.2. The η_α as a basis, antipode and coproduct

Theorem 1: Enriched monomials are a basis of QSym

Assume that 2 is invertible in $\mathbf{k}$. Then, the family $(\eta_\alpha)_{n\in\mathbb{N},\ \alpha\in\mathrm{Comp}(n)}$ **is a basis** of the $\mathbf{k}$-module QSym.

The $\mathbf{k}$-module QSym is a Hopf algebra; let S be its antipode.

Proposition 4: Antipode of enriched monomials

Let $n \in \mathbb{N}$ and $\alpha \in \mathrm{Comp}(n)$. Then, the antipode S of QSym satisfies

$$S(\eta_\alpha) = (-1)^{\ell(\alpha)} \eta_{\mathrm{rev}\,\alpha}. \tag{7}$$

Here, if $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_p)$, then $\mathrm{rev}\,\alpha := (\alpha_p, \alpha_{p-1}, \ldots, \alpha_1)$.

Theorem 2: Coproduct of enriched monomials

Consider the coproduct $\Delta : \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}$ of the Hopf algebra QSym (see [4, §5.1]). Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ be a composition. Then,

$$\Delta(\eta_\alpha) = \sum_{k=0}^{p} \eta_{(\alpha_1,\alpha_2,\ldots,\alpha_k)} \otimes \eta_{(\alpha_{k+1},\alpha_{k+2},\ldots,\alpha_p)}.$$

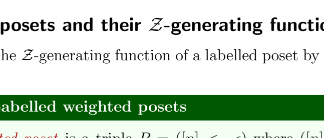
3. The product rule for the enriched monomial basis

3.1. Weighted posets and their $\mathcal{Z}$-generating functions

Now we generalize the $\mathcal{Z}$-generating function of a labelled poset by putting exponents on the $x_{|f(i)|}$'s:

Definition 2: Labelled weighted posets

A *labelled weighted poset* is a triple $P = ([n], <_P, \epsilon)$ where $([n], <_P)$ is a labelled poset and $\epsilon : [n] \longrightarrow \mathbb{P}$ is a map (called the *weight function*). In a labelled weighted poset, each node is marked with two numbers: its label $i \in [n]$ and its weight $\epsilon(i)$.



For any set $\mathcal{Z} \subseteq \mathbb{P}^\pm$, we define the *$\mathcal{Z}$-generating function* $\Gamma_\mathcal{Z}([n], <_P, \epsilon)$ of the labelled weighted poset $([n], <_P, \epsilon)$ by

$$\Gamma_\mathcal{Z}([n], <_P, \epsilon) = \sum_{f \in \mathcal{A}_\mathcal{Z}([n],<_P)} \prod_{1 \le i \le n} x_{|f(i)|}^{\epsilon(i)}. \tag{8}$$

If $\mathcal{Z} = \mathbb{P}$ or $\mathcal{Z} = \mathbb{P}^\pm$, this power series $\Gamma_\mathcal{Z}([n], <_P, \epsilon)$ lies in QSym.

Proposition 5: Decomposition of $\Gamma_\mathcal{Z}([n], <_P, \epsilon)$ into linear extensions

Let $P = ([n], <_P, \epsilon)$ be a labelled weighted poset. Let $\mathcal{L}(P)$ be the set of all linear extensions of the poset $([n], <_P)$. Then,

$$\Gamma_\mathcal{Z}([n], <_P, \epsilon) = \sum_{([n],<_L) \in \mathcal{L}(P)} \Gamma_\mathcal{Z}([n], <_L, \epsilon). \tag{9}$$

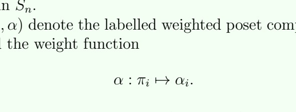
3.2. Universal quasisymmetric functions

If a labelled poset $P = ([n], <_P)$ is totally ordered (i.e., $<_P$ is total), it can be described by a permutation $\pi \in S_n$. In this case, we give its $\mathcal{Z}$-generating function $\Gamma_\mathcal{Z}([n], <_P, \epsilon)$ a new name:

Definition 3: Universal quasisymmetric functions

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a composition. Let $\pi = \pi_1 \ldots \pi_n$ (in one-line notation) be a permutation in S_n.
Let $P_{\pi,\alpha} = ([n], <_\pi, \alpha)$ denote the labelled weighted poset composed of the labelled poset $([n], <_\pi)$ and the weight function

$$\alpha : \pi_i \mapsto \alpha_i.$$



We define the *universal quasisymmetric function* $U^\mathcal{Z}_{\pi,\alpha}$ as the generating function

$$U^\mathcal{Z}_{\pi,\alpha} = \Gamma_\mathcal{Z}([n], <_\pi, \alpha). \tag{10}$$

Thus, by (9), each $\mathcal{Z}$-generating function $\Gamma_\mathcal{Z}([n], <_P, \epsilon)$ is a sum of $U^\mathcal{Z}_{\pi,\alpha}$s.
The $U^\mathcal{Z}_{\pi,\alpha}$s include, as particular cases, all four bases we have seen so far (thus the "universal"):

Proposition 6: Specialisation of universal quasisymmetric functions

Let $n \in \mathbb{N}$. Let id_n and $\overline{id_n}$ denote the two permutations in S_n given by $id_n = 1\ 2\ 3 \ldots n$ and $\overline{id_n} = n\ n-1\ n-2 \ldots 1$ (in one-line notation). Let (1^n) be the composition $(1, 1, \ldots, 1)$ of n. Then:

- For any $\pi \in S_n$, we have

$$U^{\mathbb{P}}_{\pi,(1^n)} = L_\pi, \qquad U^{\mathbb{P}^\pm}_{\pi,(1^n)} = K_\pi.$$

- For any composition α with n entries, we have

$$U^{\mathbb{P}}_{\overline{id_n},\alpha} = M_\alpha, \qquad U^{\mathbb{P}^\pm}_{id_n,\alpha} = \eta_\alpha \tag{11}$$

(where we identify α with the appropriate weight function as in the definition above).

3.3. Product rule

Theorem 3: Product of universal quasisymmetric functions

Let $\mathcal{Z}$ be a subset of $\mathbb{P}^\pm$. Let π and σ be two permutations in S_n and S_m, and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_m)$ be two compositions with n and m entries. The product of two universal quasisymmetric functions is given by

$$U^\mathcal{Z}_{\pi,\alpha} U^\mathcal{Z}_{\sigma,\beta} = \sum_{(\tau,\gamma) \in (\pi,\alpha) ⧢ (\sigma,\beta)} U^\mathcal{Z}_{\tau,\gamma}, \tag{12}$$

where the sum is over all ways to shuffle π with $m + \sigma$ and – using the same permutation – shuffle α with β.

This follows from (9) and from the obvious product rule for $\Gamma_\mathcal{Z}([n], <_\pi, \alpha)$'s:

$$\Gamma_\mathcal{Z}([n], <_P, \alpha) \cdot \Gamma_\mathcal{Z}([m], <_Q, \beta) = \Gamma_\mathcal{Z}([n+m], <_{PQ}, \alpha \sqcup \beta), \tag{13}$$

where $<_{PQ}$ is the disjoint union of the relation $<_P$ on $[n]$ and the (shifted-by-n) relation $<_Q$ on $[n+1, n+m]$.

3.4. Product of enriched monomials

Let α and β be two compositions with n and m entries. Equations (11) and (12) imply:

$$\eta_\alpha \eta_\beta = U^{\mathbb{P}^\pm}_{id_n,\alpha} U^{\mathbb{P}^\pm}_{id_m,\beta} = \sum_{(\tau,\gamma) \in (id_n,\alpha) ⧢ (id_m,\beta)} U^{\mathbb{P}^\pm}_{\tau,\gamma}. \tag{14}$$

We shall now rewrite this as an alternating sum of η_γs.

Definition 4: Composition reduction

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a composition with n entries. For any $1 < i < n$, let $\alpha^{\downarrow\downarrow i}$ denote the following composition with $n - 2$ entries:

$$\alpha^{\downarrow\downarrow i} = (\alpha_1, \ldots, \alpha_{i-2}, \alpha_{i-1} + \alpha_i + \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_n).$$

Furthermore, if $I = \{i_1 < i_2 < \cdots < i_k\} \subseteq [2, n-1]$ is a set containing no two consecutive integers, then we set

$$\alpha^{\downarrow\downarrow I} = \left(\left(\cdots\left(\alpha^{\downarrow i_k}\right)\cdots\right)^{\downarrow i_2}\right)^{\downarrow i_1}.$$

(In particular, $\alpha^{\downarrow\downarrow\varnothing} = \alpha$.)

Example: If $\alpha = (2,1,4,3,2)$, then $\alpha^{\downarrow\downarrow 3} = (2,8,2)$ and $\alpha^{\downarrow\downarrow\{2,4\}} = (12)$.

Theorem 4: Universal quasisymmetric functions to enriched monomials

Let α be a composition with n entries and π a permutation in S_n. We have

$$U^{\mathbb{P}^\pm}_{\pi,\alpha} = \sum_{I \subseteq \mathrm{Peak}(\pi)} (-1)^{|I|} \eta_{\alpha^{\downarrow\downarrow I}}. \tag{15}$$

Proof (sketch). From the definition of $U^{\mathbb{P}^\pm}_{\pi,\alpha}$, one easily obtains

$$U^{\mathbb{P}^\pm}_{\pi,\alpha} = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n;\\ j \in \mathrm{Peak}(\pi) \Longrightarrow \neg(i_{j-1} = i_j = i_{j+1})}} 2^{|\{i_1,i_2,\ldots,i_n\}|} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_n}^{\alpha_n}.$$

By the inclusion-exclusion principle, this can be rewritten as

$$U^{\mathbb{P}^\pm}_{\pi,\alpha} = \sum_{I \subseteq \mathrm{Peak}(\pi)} (-1)^{|I|} \underbrace{\sum_{\substack{i_1 \le i_2 \le \cdots \le i_n;\\ j \in I \Longrightarrow i_{j-1} = i_j = i_{j+1}}} 2^{|\{i_1,i_2,\ldots,i_n\}|} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_n}^{\alpha_n}}_{= \eta_{\alpha^{\downarrow\downarrow I}}}. \qquad \square$$

Theorem 5: Product rule for enriched monomials

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_m)$ be two compositions. Given a composition γ obtained by shuffling α and β (we shall denote this by $\gamma \in \alpha ⧢ \beta$), let $S_\beta(\gamma)$ be the set of the positions of the entries of β in γ. Denote further $S_\beta(\gamma) - 1 = \{i - 1 \mid i \in S_\beta(\gamma)\}$. Then,

$$\eta_\alpha \eta_\beta = \sum_{\substack{\gamma \in \alpha ⧢ \beta;\\ I \subseteq \left(S_\beta(\gamma) \setminus (S_\beta(\gamma) - 1)\right) \setminus \{1\}}} (-1)^{|I|} \eta_{\gamma^{\downarrow\downarrow I}}. \tag{16}$$

The sum ranges not over compositions γ but over ways to shuffle α with β. Thus, the same γ can appear in several addends of the sum.
As an example, one has

$$\begin{aligned}\eta_{(1,1)}\eta_{(2,3)} &= \eta_{(1,1,2,3)} + \eta_{(1,2,1,3)} - \eta_{(4,3)} + \eta_{(2,1,1,3)} + \eta_{(1,2,3,1)}\\ &\quad - \eta_{(1,6)} + \eta_{(2,1,3,1)} - \eta_{(2,5)} + \eta_{(2,3,1,1)} - \eta_{(6,1)},\\ \eta_{(1,2)}\eta_{(2)} &= \eta_{(2,1,2)} + 2\eta_{(1,2,2)} - \eta_{(5)}.\end{aligned}$$

Proof (sketched). Recall (14), and rewrite the right-hand side using (15). Let $(\tau, \gamma) \in (id_n, \alpha) ⧢ (id_m, \beta)$. The index i belongs to $\mathrm{Peak}(\tau)$ if and only if τ_i is a letter of $n + id_m$ and τ_{i+1} is a letter of id_n and we have $i > 1$. That is, if and only if i is the index of an entry of β in γ and $i+1$ is the index of an entry of α. Thus, $\mathrm{Peak}(\tau) = (S_\beta(\gamma) \setminus (S_\beta(\gamma) - 1)) \setminus \{1\}$. Therefore, we obtain (16). $\square$

The product formula (16) generalizes a result of Hsiao [5].

3.5. Antipodes of $\mathcal{Z}$-generating functions (NEW!)

The formula (7) for the antipode of η_α can be generalized along the lines of [2]:

Theorem 6: Antipode of $\Gamma_\mathcal{Z}([n], <_P, \epsilon)$

Let $P = ([n], <_P, \epsilon)$ be a labelled weighted poset. Assume that $\mathcal{Z} = \mathbb{P}$ or $\mathcal{Z} = \mathbb{P}^\pm$. Let S be the antipode of the Hopf algebra QSym. Then,

$$S\left(\Gamma_\mathcal{Z}([n], <_P, \epsilon)\right) = (-1)^n \Gamma_\mathcal{Z}([n], >_P, \epsilon), \tag{17}$$

where $>_P$ is the opposite relation to $<_P$.

For $\mathcal{Z} = \mathbb{P}$, this is [2, Corollary 9], generalizing the known antipode formulas for Schur (and dual immaculate Schur) functions.

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