Counting Linear Extensions of Posets via Determinants of Hooks

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Joint work with
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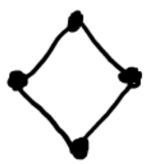
Partially Ordered Sets (Posets)

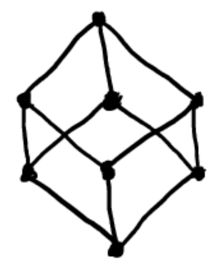
A partially ordered set P is a set of elements P and a relation $\leq_P$ such that

(1) $x \leq_P x$

(2) $x \leq_P y$ and $y \leq_P x \implies x = y$

(3) $x \leq_P y$ and $y \leq_P z \Rightarrow x \leq_P z$





Linear Extensions

Let P be a poset on n elements.
A linear extension of P is a bijection $f: P \to [n]$ where
$$x \le_P y \implies f(x) \le f(y).$$

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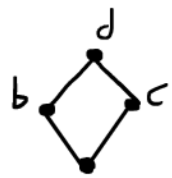
Example:



P

has linear extensions

Linear Extensions

Counting linear extensions has many uses

- Counting permutations
- Volumes of Polytopes
- Bijections to many objects
- Standard/Skew tableaux

Linear Extensions

Q: Given a poset, can we efficiently count the number of linear extensions?

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However, there are many families of posets that admit closed formulas or efficient computation.

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Families of Posets

1) Zigzags / fences

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ie $(a_1, a_2, a_3, \dots)$ such that $a_i < a_{i+1} > a_{i+2}$ for all i even

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ie $(a_1, a_2, a_3, \dots)$ such that $a_i < a_{i+1} > a_{i+2}$ for all i even

<u>Theorem</u> (André, 1881): Let E_n be the number of alternating permutations on n elements.

$$\sum_{i=0}^{\infty} E_i \frac{x^i}{i!} = \sec x + \tan x$$

Families of Posets

2) Descent Sets / Ribbon Posets

Let $S = \{s_1, \ldots, s_k\} \subset \{1, \ldots, n-1\}$ with $s_1 < s_2 < \cdots < s_k$

A permutation $\sigma \in S_n$ with descent set S has $\sigma_{s_i} > \sigma_{s_i+1}$ for all i.

Example: $(5,4,2,3,1)$ has $S = \{1,2,4\}$

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Q: How many permutations have a descent set S?

Equivalently, count the number of linear extensions of the ribbon.

$P_S =$  $S = \{\underline{1}, \underline{2}, \underline{4}\}$ $\underline{n = 5}$

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Families of Posets

2) Descent Sets / Ribbon Posets

A: <u>Theorem (MacMahon, 1915)</u>: Given n elements and descent set

$S = \{ s_1, s_2, \ldots, s_k \}$

$$e(P_S) = n! \cdot \det \left| \frac{1}{(s_{j+1} - s_i)!} \right|_{0 \le i,j \le k}$$

where $e(P_S)$ denotes the number of linear extensions, $s_0 = 0$, $s_{k+1} = n$

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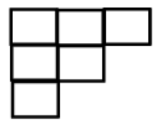
Families of Posets

3) Young Tableax

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be a partition of n.

$\lambda = (3, 2, 1)$



$\lambda_1 = 3$
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partition of 6 of shape λ.

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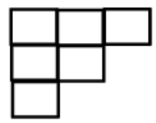

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Standard Young Tableaux:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$$

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Families of Posets

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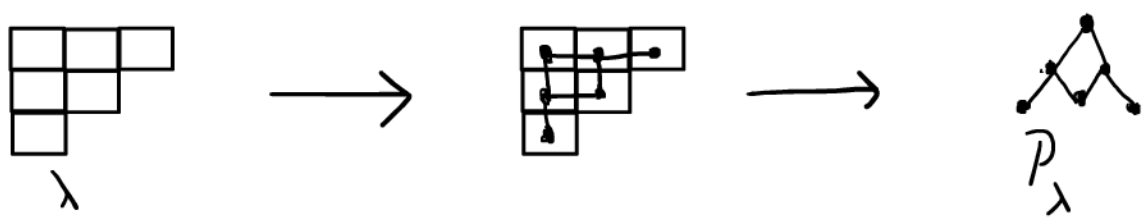

$$e(P_\lambda) = f^\lambda := \#\text{ standard Young tableaux}$$

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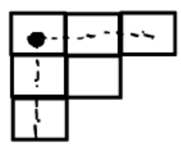
Families of Posets

3) Young Tableax

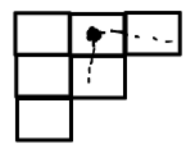
Theorem (Hook Length Formula, FRT54):

$$e(P_\lambda) = n! \cdot \prod_{p \in P_\lambda} \frac{1}{h_p}, \quad h_p \text{ the hook length of } p.$$

Example:



$h_{(1,1)} = 5$



$h_{(2,1)} = 3$

$\Rightarrow e($  $) = \frac{6!}{5 \cdot 3 \cdot 3 \cdot 1 \cdot 1 \cdot 1} = 16$

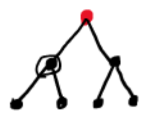
Families of Posets

4) Rooted Trees

<u>Theorem (Hook Length Formula, Knuth):</u> Let P be a rooted tree poset.

$$e(P) = n! \cdot \prod_{p \in P} \frac{1}{h_p}, \quad h_p \text{ the hook length of } p.$$

Example:



$h_{root} = 7$

$\Rightarrow$ e() $= \dfrac{7!}{7 \cdot 3 \cdot 3} = 80$

Families of Posets

Other Families:

- ▷ Skew Tableaux (determinant formula)
- ▷ Tree posets ($O(n^2)$ algorithm)
- ▷ d-complete posets (Hook length formula)

Our Results

Goal: Find a closed form for new classes of posets.

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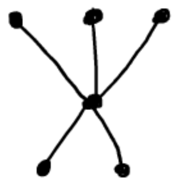
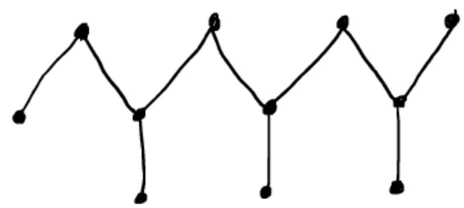
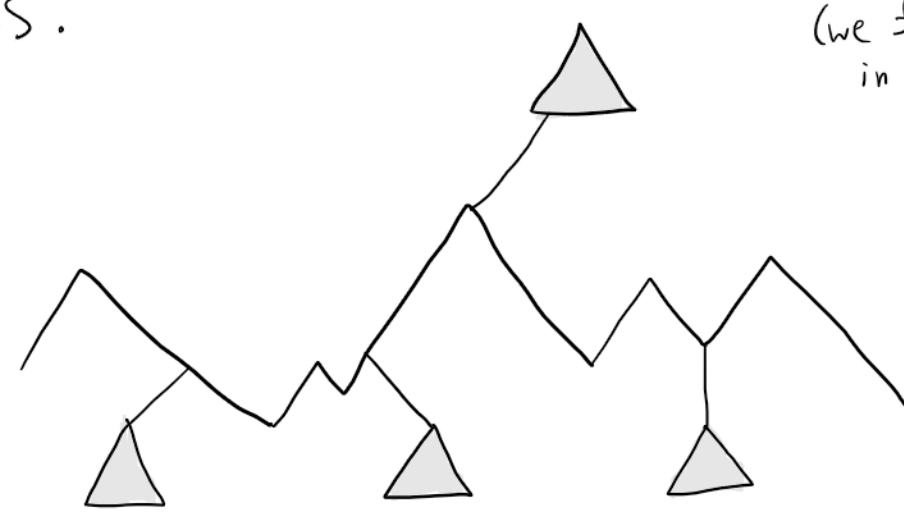




Our Results

Mobiles:



*  are d-complete

(we focus on rooted trees in this talk)

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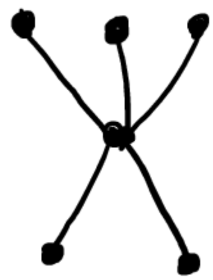
Inclusion-Exclusion by Folding

$P =$ 

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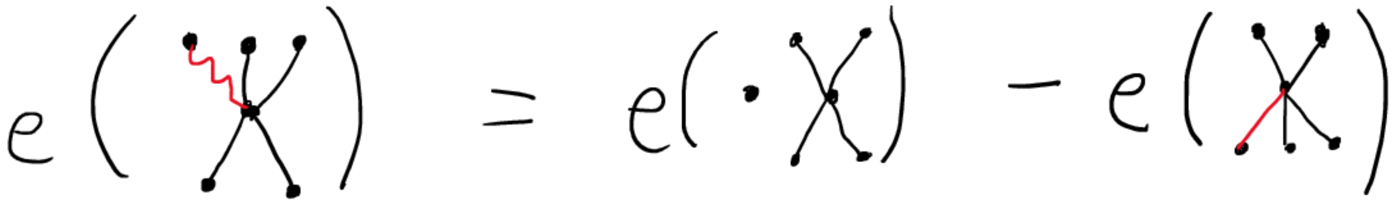
Inclusion-Exclusion by Folding



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Inclusion-Exclusion by Folding

For a set of edges $F \subset \lessdot_P$ and $S \subseteq F$, let $P_{S,F}$ denote the poset where we remove F and add back the folded S.

ie.  F $S = \{\ \}$, $P_{S,F}$ =  .

P

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$S = \{$  $\}$, $P_{S,F} =$  .

Lemma: P poset. $F \subset \lessdot_P$, then

$$e(P) = \sum_{S \subset F} (-1)^{|S|} e(P_{S,F})$$

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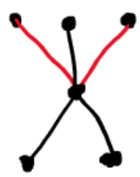
Inclusion-Exclusion by Folding

Lemma: P poset. $F \subset \lessdot_P$, then

$$e(P) = \sum_{S \subset F} (-1)^{|S|} e(P_{S,F})$$

Example:

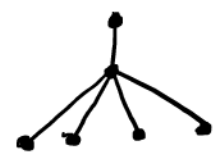
$e\Big($  $\Big) = e\Big($  $\Big) - 2e\Big($  $\Big)$

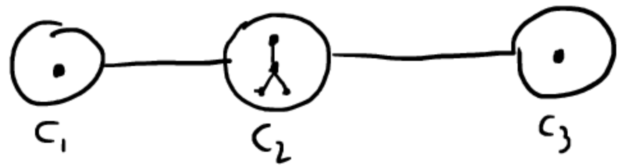
$+\ e\Big($  $\Big)$

Connected Component Graph

Given P and folds F, removing the folds creates connected components.

P: 

, P_F: 

$C(P_F)$ = 

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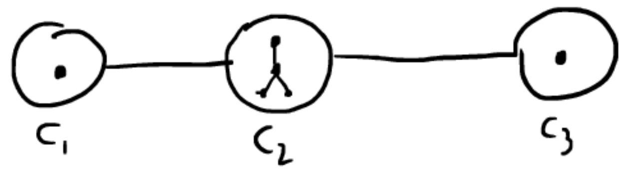
$C(P_F) =$ 

We want when $C(P_F)$ is a path

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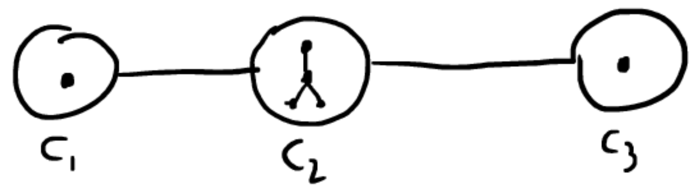
Component Array

$\mathcal{P}$: 

$C(\mathcal{P}_F) =$ 

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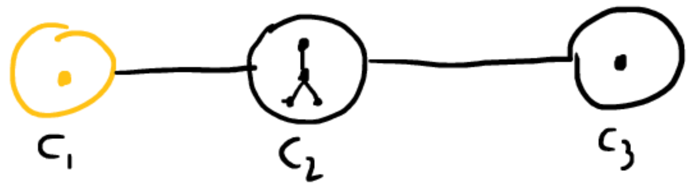
$C(\mathcal{P}_F)$ = 

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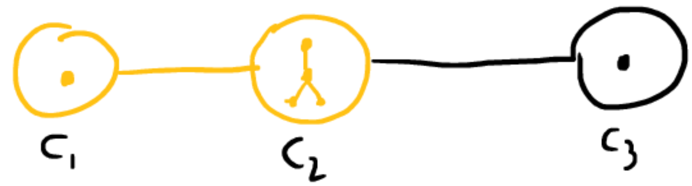
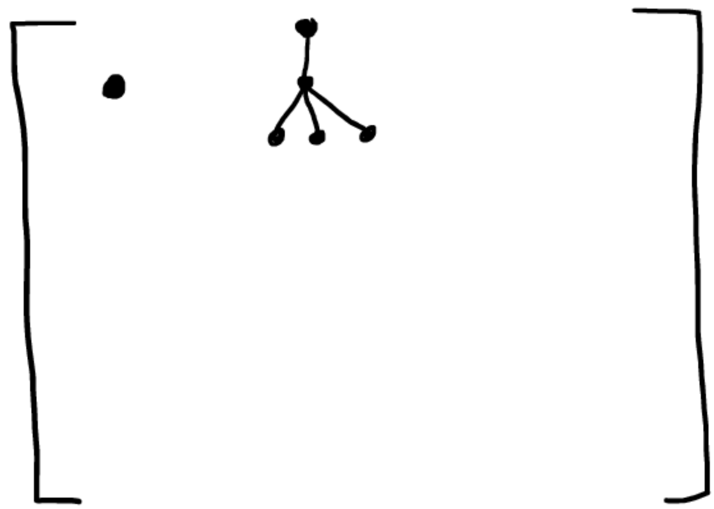
$C(\mathcal{P}_F) =$ 



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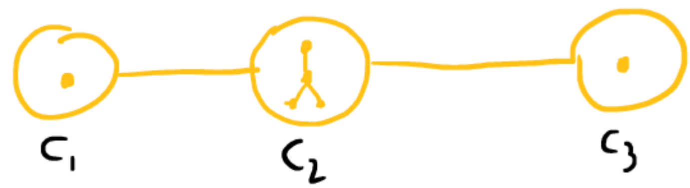
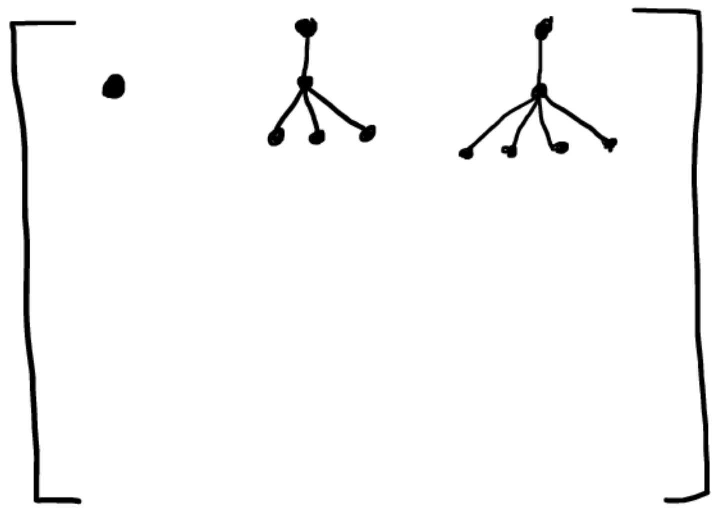
$C(\mathcal{P}_F) =$ 



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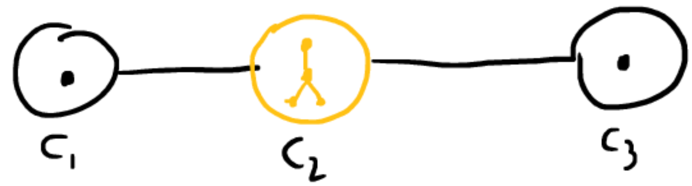
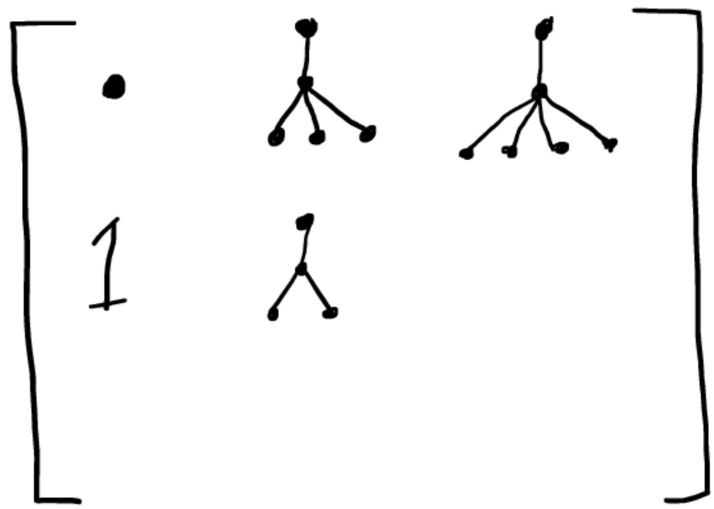
$C(\mathcal{P}_F) =$ 



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Component Array

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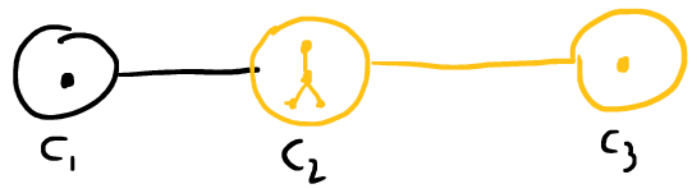
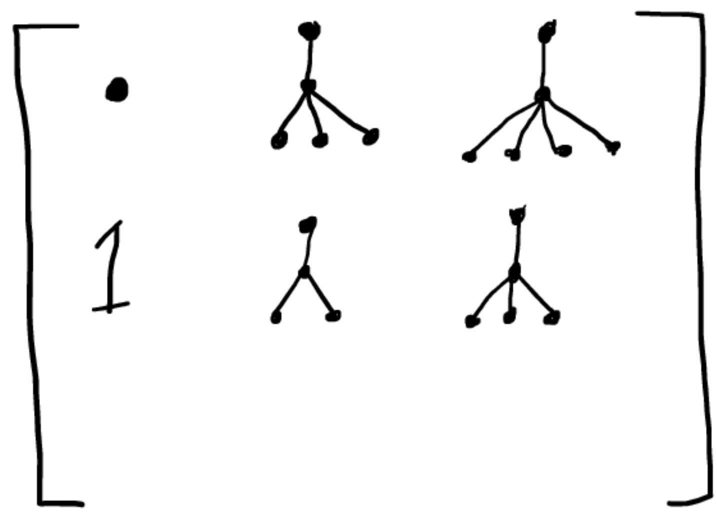
$C(\mathcal{P}_F) =$ 



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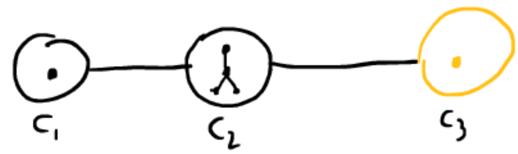
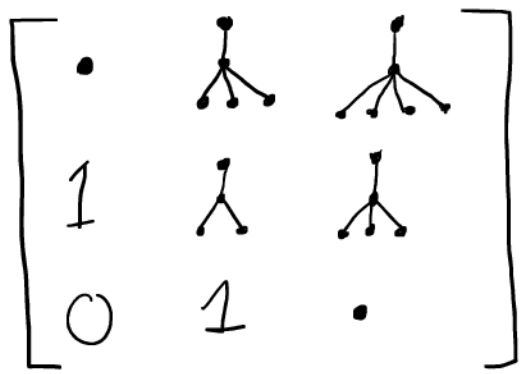
Component Array

P: 

$C(P_F) =$ 

$M_\sigma(P_F) :=$ 

where $M_{i,j}$ is the subposet of P_F on components $C_i \dots C_j$ for $j \geq i$

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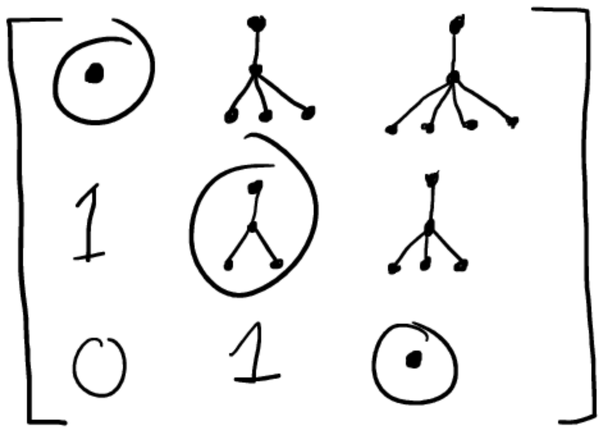
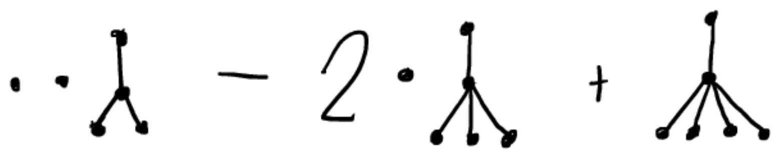
Component Array

$M_\sigma(P_F) :=$



Alternating Formula:



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Component Array

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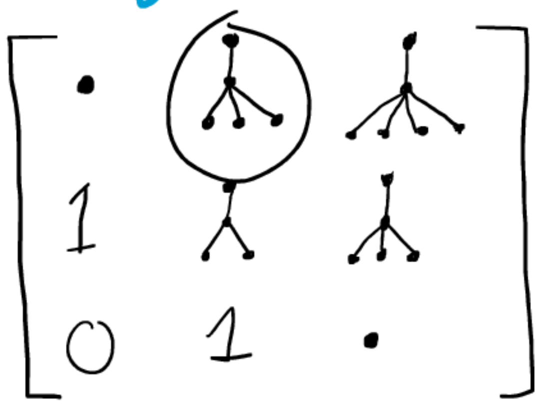

$$\bar{e}(M_\sigma(P_F)) = \begin{bmatrix} \frac{1}{1} & \frac{1}{5\cdot 4} & \frac{1}{6\cdot 5} \\ 1 & \frac{1}{4\cdot 3} & \frac{1}{5\cdot 4} \\ 0 & 1 & \frac{1}{1} \end{bmatrix} \qquad \bar{e}(M_\sigma(P_F))_{i,j} = \begin{cases} 0 & j < i-1 \\ 1 & j = i-1 \\ \prod_{p \in M_{i,j}} h_p & \text{otw} \end{cases}$$

Determinant Formula for Mobiles

Theorem (GGMM): Let P be a tree poset on n elements with folds $F \subset <_P$ selected so that P_F is a rooted tree and $C(P_F)$ is a path. Then

$$e(P) = n! \det(\bar{e}(M(P_F)))$$

Determinant Formula for Mobiles

$e(P) = n!\,\det(\bar{e}(M(P_F)))$

Proof:

1) Write alternating formula for $e(P)$ with F.

2) Convert the alternating formula to a determinant.

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Determinant Formula for Mobiles

$e(P) = n!\, \det(\overline{e}(M(P_F)))$

Proof: 2) Convert the alternating formula to a determinant.

Lemma (Stanley, EC): Let g be any function on $[0, k+1] \times [0, k+1]$ such that $g(i,i) = 1$, $g(i,j) = 0$ for $j < i$.

$$D_k := \sum_{1 \le i_1 \le \cdots \le i_j \le k} (-1)^{k-j} g(0, i_1) g(i_1, i_2) \cdots g(i_j, k+1),$$

$$\det\left([g(i, j+1)]_{i,j} \right) = D_k$$

Determinant Formula for Mobiles

$e(P) = n! \det(\bar{e}(M(P_F)))$

Proof:

Apply lemma to alternating formula with

$$g(i, j+1) = \bar{e}(M(P_F))_{i,j}$$

$\square$

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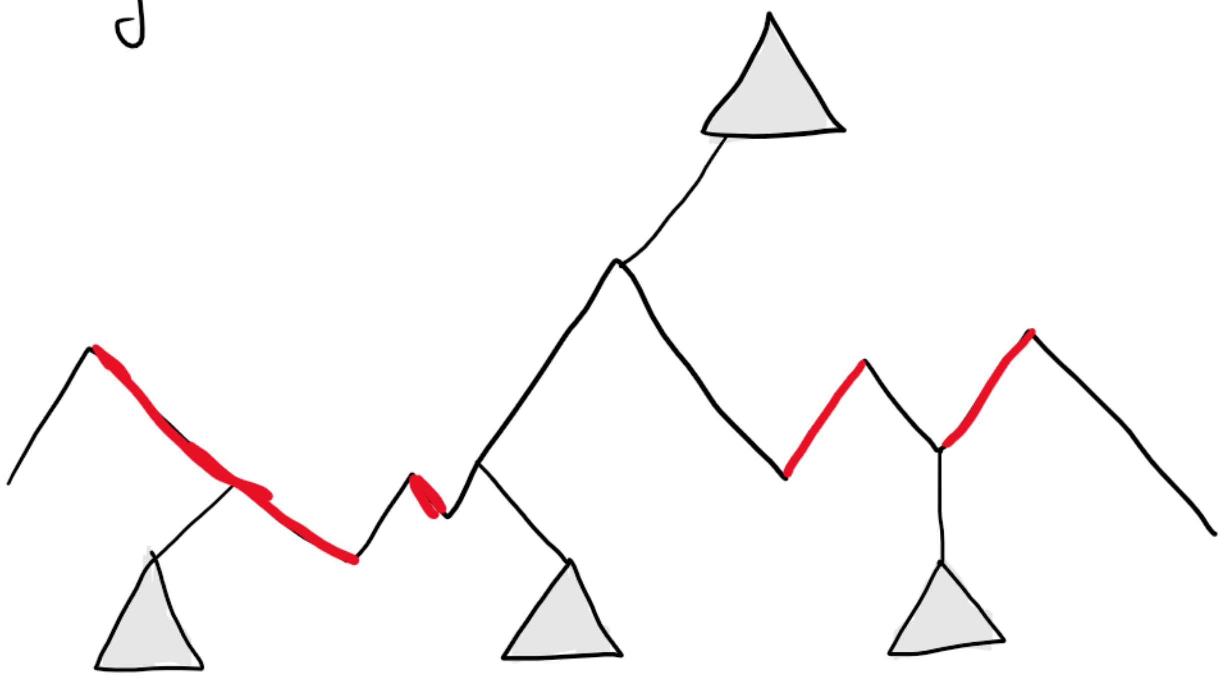
Determinant Formula for Mobiles

Folding a Mobile:



Thanks!

Check out GaYee Park's talk in this poster session, also about mobile posets!