Equidistributions of mesh patterns of length two

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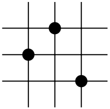
FPSAC Online, 18 Jan, 2022

Classical pattern

Let S_n be the set of all permutations of length n. A (classical permutation) pattern is a permutation $\tau \in S_n$. We could draw the pattern $231 \in S_3$ as follows, where the horizontal lines represent the values and the vertical lines denote the positions in the pattern.

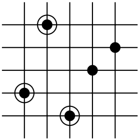


Classical pattern

We say that a pattern p *occurs* in a permutation $\pi \in S_n$ if there is a subsequence of π whose letters are in the same relative order of size as the letters of p. This sequence is called an *occurrence* of the pattern p in the permutation π. If a pattern occurs in a permutation we say that the permutation *contains* the pattern.

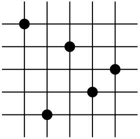
Example

The permutation 25134 contains the pattern 231 as the subsequence 251. The diagram below shows the permutation where points corresponding to the occurrence of the pattern have been circled.

25134 = 

Pattern avoidance

A permutation that does not contain a pattern is said to *avoid* the pattern. For example, a permutation $\pi \in S_n$ avoids the pattern 231 if there do not exist $1 \leq i < j < k \leq n$ with $\pi(k) < \pi(i) < \pi(j)$. An example of a permutation that avoids the pattern 231 is the permutation 51423.

$51423 =$ 

Given a pattern p we let $S_n(p)$ be the set of permutations of length n that avoid p.

Wilf-class

Two patterns, p and q, are said to be *Wilf-equivalent* if for any $n \geq 0$, the sizes of the sets $S_n(p)$ and $S_n(q)$ are equal.
A *Wilf-class* is a maximal set of patterns (necessarily of the same length) that are all Wilf-equivalent.
The process of sorting patterns into classes by Wilf-equivalence is called *Wilf-classification*.

Wilf-class

- Classical patterns of length 3 were Wilf-classified by Knuth, who showed that the number of permutations avoiding each classical pattern of length 3 is given by the Catalan numbers.
- Permutations avoiding more than one pattern have also been studied. Simion and Schmidt Wilf-classified all sets of classical patterns of length 3.

D. E. Knuth, The Art of Computer Programming, volume 1: Fundamental Algorithms, Addison-Wesley Publishing Co., 1973.

Rodica Simion and Frank W. Schmidt, Restricted permutations, European J. Combin. 6 (1985), no. 4, 383–406.

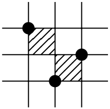
Mesh pattern

Mesh patterns where first introduced by Brändén and Claesson (2011), as a further extension of bivincular patterns.
A pair (τ, R), where τ is a permutation in S_k and R is a subset of $[\![0, k]\!] \times [\![0, k]\!]$, where $[\![0, k]\!]$ denotes the interval of the integers from 0 to k, is a *mesh pattern* of length k.

P. Brändén and A. Claesson, Mesh patterns and the expansion of permutation statistics as sums of permutation patterns, Electron. J. Combin. 18 (2011), no. 2, Paper 5, 14.

Mesh pattern

Let (i,j) denote the box whose corners have coordinates $(i,j),(i,j+1),(i+1,j+1)$ and $(i+1,j)$. An example of a mesh pattern is the classical pattern 312 along with $R=\{(1,2),(2,1)\}$. We draw this by shading the boxes in R



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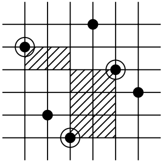
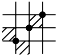
Mesh pattern

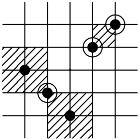
The permutation 521643 contains this pattern, see below

521643 = 

The permutation has an occurrence of the mesh pattern as the subsequence 514, since it forms the classical pattern 312 and there are no points in the shaded areas.

Mesh pattern

Let's now look at the permutation $\pi = 32145$. This permutation avoids the pattern $(123, \{(0,1),(1,0),(2,2)\}) =$ , because for all occurrences of the classical pattern 123 there is at least one point in at least one of the shaded boxes. For example, the subsequence 245 in π is an occurrence of the classical pattern 123 but not of the mesh pattern since the point representing 1 is in one of the shaded areas. This can be seen on the following diagram.

$32145 =$ 

Recent works related to the study of mesh pattern

- J.-L. Baril, Classical sequences revisited with permutations avoiding dotted pattern, Electron. J. Combin., 18(1) (2011), P178.
- M. Jones, S. Kitaev, and J. Remmel, Frame patterns in n-cycles, Discr. Math., 338 (2015), 1197-1215.
- S. Kitaev and J. Liese, Harmonic numbers, Catalan's triangle and mesh patterns, Discr. Math., 313 (2013), 1515-1531.
- S. Kitaev and J. Remmel, Quadrant marked mesh patterns in alternating permutations, Séminaire Lotharingien de Combinatoire, B68a (2012).
- B. E. Tenner, Mesh patterns with superfluous mesh, Adv. Appl. Math., 51 (2013), 606-618.

Recent works related to the study of mesh pattern

Hilmarsson et al. studied the Wilf-classification of mesh patterns of length 2.
Kitaev and Zhang further studied the distributions of mesh patterns considered by Hilmarssson et al. by giving 27 distribution results.

I. Hilmarsson, I. Jónsdóttir, S. Sigurdardóttir, L. Vidarsdóttir, and H. Ulfarsson, Wilf-classification of mesh patterns of short length, Electr. J. Combin., 22(4) (2015), #P4.13.

S. Kiatev, P. B. Zhang, Distributions of mesh patterns of short lengths. Adv. in Appl. Math. 110 (2019), 1–32.

Kitaev and Zhang's Conjecture

For a pattern p and a permutation π, we let $p(\pi)$ denote the number of occurrences of p in π.

Conjecture (Kitaev and Zhang)

We have

$$\sum_{n\geq 0} t^n \sum_{\pi\in S_n} y^{(\pi)} = \cfrac{1}{1-\cfrac{\alpha_1 t}{1-\cfrac{\alpha_2 t}{1-\cdots}}} \tag{1a}$$

with coefficients

$$\alpha_{2k-1} = k, \quad \alpha_{2k} = y + k - 1. \tag{1b}$$

Permutation Statistics

Given a permutation $\pi \in S_n$, an index $i \in [n]$ (or a value $\pi(i) \in [n]$) is called

- an *excedance* if $\pi(i) > i$;
- an *inversion* if $\pi(j) > \pi(i)$ for $1 \leq j < i$; in other words, an inversion of π is one occurrence of pattern  of π;

We denote the number of excedances, inversions in π by $\mathrm{exc}(\pi)$, $\mathrm{inv}(\pi)$, respectively.

Permutation Statistics

Given a permutation $\pi \in S_n$, an index $i \in [n]$ (or a value $\pi(i) \in [n]$) is called

- a *record* (rec) (or *left-to-right maximum*) if $\pi(j) < \pi(i)$ for all $j < i$ [note in particular that the index 1 is always a record and that the value n is always a record]; in other words, a record of π is one occurrence of pattern  of π;
- an *antirecord* (arec) (or *right-to-left minimum*) if $\pi(j) > \pi(i)$ for all $j > i$ [note in particular that the index n is always an antirecord and that the value 1 is always an antirecord]; in other words, an antirecord of π is one occurrence of pattern  of π;

We denote the number of records, antirecords in π by $\mathrm{rec}(\pi)$, $\mathrm{arec}(\pi)$, respectively.

Permutation Statistics

Given a permutation $\pi \in S_n$, an index $i \in [n]$ (or a value $\pi(i) \in [n]$) is called an *exclusive antirecord* (earec) if it is an antirecord and not also a record. We denote the number of exclusive antirecords in π by $\mathrm{earec}(\pi)$.

Lemma

For $\pi \in S_n$, we have

$$\mathrm{earec}(\pi) = $$  $(\pi) =$  $(\pi) =$  $(\pi) =$  (π)

Proof.

In the rook placement representation of a permutation $\pi \in S_n$ the rook $y = (i, \pi(i))$ is an exclusive antirecord iff there is a another rook $x = (j, \pi(j))$ at left of y, i.e., $j < i$ and higher than x, i.e., $\pi(j) > \pi(i)$. Hence there are four unique choices for such a rook x: the *highest*, *lowest*, *farthest* and *nearest*. $\square$

Permutation Statistics

Given a permutation $\pi \in S_n$, an index $i \in [n]$ (or a value $\pi(i) \in [n]$) is called

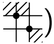
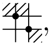
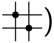
- an *exclusive record* (erec) if it is a record and not also an antirecord; in other words, an exclusive record of π is one occurrence of pattern  of π.
- a *record-antirecord* (rar) (or *pivot*) if it is both a record and an antirecord; in other words, a record-antirecord of π is one occurrence of pattern  of π.

We denote the number of exclusive records and record-antirecords in π by $\mathrm{erec}(\pi)$ and $\mathrm{rar}(\pi)$, respectively.

Proposition

$\mathrm{erec}(\pi) =$  $(\pi) =$  $(\pi) =$  $(\pi) =$  (π).

Generalized Eulerian polynomials

Dumont and Kreweras gave the joint distribution of (, ), Zeng gave the joint distribution of (, , ). Recently Sokal and Zeng proved much more general results.

- D. Dumont, G. Kreweras, Sur le développement d'une fraction continue liée à la série hypergéométrique et son interprétation en termes de records et anti-records dans les permutations, European J. Combin. 9, 27–32 (1988).
- A. D. Sokal, J. Zeng, Some multivariate master polynomials for permutations, set partitions, and perfect matchings, and their continued fractions, arXiv preprint (2020). (arXiv:2003.08192)
- J. Zeng, Records, antirecords et permutations discordantes, European J. Combin. 10, 103–109 (1989).

Generalized Eulerian polynomials

Define the generating function of the generalized Eulerian polynomials

$$F(x,y,z,v,q;t)=\sum_{n=0}^{\infty}t^n\sum_{\sigma\in\mathfrak{S}_n}x^{\mathrm{arec}(\sigma)}y^{\mathrm{erec}(\sigma)}z^{\mathrm{rar}(\sigma)}v^{\mathsf{exc}(\sigma)}q^{\mathsf{inv}(\sigma)}.$$

Theorem (Han-Zeng)

We have

$$F(x,y,z,v,q;t) = \frac{F(x,y,1,v,q;t)}{1 + x(1-z)tF(x,y,1,v,q;t)}, \tag{2a}$$

where

$$F(x,y,1,v,q;t) = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}} \tag{2b}$$

with coefficients

$$\alpha_{2k-1} = q^{k-1}(x + q + q^2 + \cdots + q^{k-1}) \tag{2c}$$

$$\alpha_{2k} = q^k v(y + q + q^2 + \cdots + q^{k-1}). \tag{2d}$$

symmetry operations

The first known operations are the symmetries reverse, complement and inverse. For a given mesh pattern (τ, R) of length n, we define

$$(\tau, R)^r = (\tau^r, R^r), \quad (\tau, R)^c = (\tau^c, R^c), \quad (\tau, R)^{-1} = (\tau^{-1}, R^{-1}),$$

where τ^r is the usual reverse of the permutation τ, τ^c the usual complement, τ^{-1} the usual inverse, and

$$\begin{aligned}
\tau^r &:= \tau(n) \cdots \tau(2)\tau(1), \\
\tau^c &:= (n+1-\tau(1))(n+1-\tau(2)) \cdots (n+1-\tau(n)), \\
\tau^{-1} &:= \tau^{-1}(1)\tau^{-1}(2) \cdots \tau^{-1}(n), \\
R^r &= \{(n-x, y) \colon (x, y) \in R\}, \\
R^c &= \{(x, n-y) \colon (x, y) \in R\}, \\
R^{-1} &= \{(y, x) \colon (x, y) \in R\}.
\end{aligned}$$

Example

The following figure is an example of the use of these symmetries on the pattern $p = (312, \{(0,1), (1,3), (2,2)\})$.

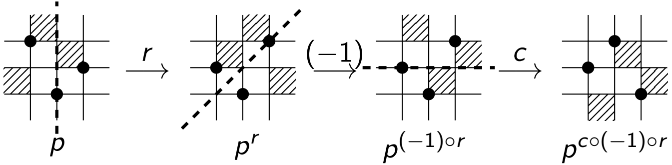


Figure: Several symmetries of a mesh pattern

Note that reverse is a reflection around the vertical center line, complement is a reflection around the horizontal center line and inverse is the reflection around the southwest to northeast diagonal.

Main result I

From the continued fraction, the conjecture becomes

$(\pi) = \mathrm{erec}(\Phi(\pi))$

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Theorem (Han-Zeng)

There exists an involution Φ on S_n such that for $\pi \in S_n$,

$$(\ $$$$,$$ $$,$$ $$)\pi = (\ $$$$,$$ $$,$$ $$)\Phi(\pi).$$

Two operations

For $\pi \in S_n$ let $\mathrm{AREC}(\pi) = (i_1, i_2, \ldots, i_l)$ be the sequence of antirecord positions of π from left to right. So $\pi(i_1) = 1$, $i_1 < \cdots < i_l$ and $i_l = n$.

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$$\varphi_1^{(i_k)} : \pi \mapsto \pi' \tag{3a}$$

$$\varphi_2^{(i_k)} : \pi \mapsto \pi'' \tag{3b}$$

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- let $w' = w'_1 \ldots w'_r$ be the word obtained by substituting the jth largest letter with the jth smallest letter in w for $j = 1, \ldots, r$;
- let π' (resp. π'') be the word obtained by replacing w_j with w'_j for $j = 1, \ldots, r$ in π.

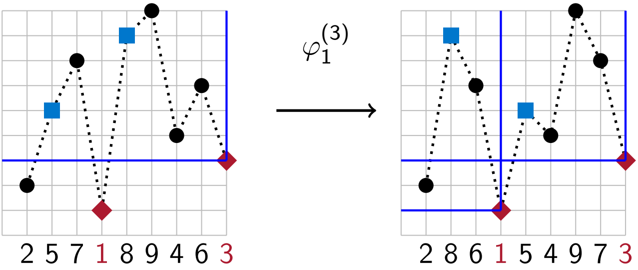
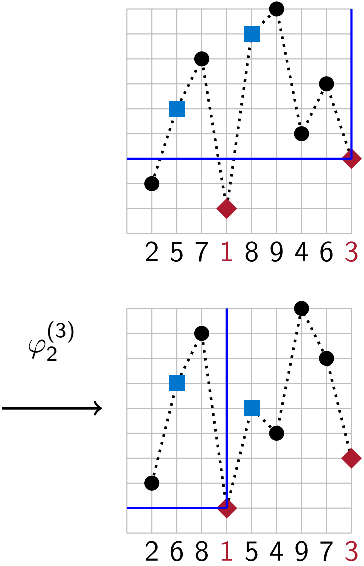


Figure: The involution Φ on the permutation 257189463



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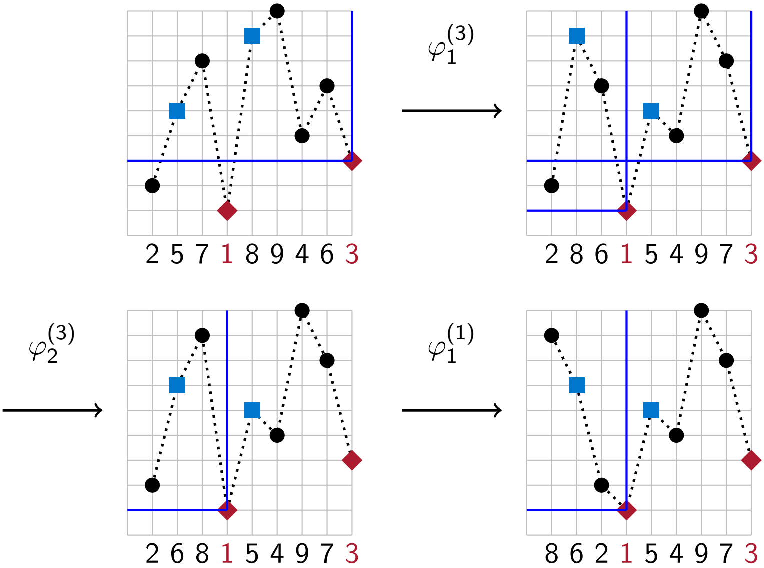

Figure: The involution Φ on the permutation 257189463

The operation Φ

Let $\pi \in S_n$ with sequence of antirecord positions $\mathrm{AREC}(\pi) = (i_1, i_2, \ldots, i_l)$. We define the operation Φ on π by

$$\Phi(\pi) = \varphi^{(i_1)} \circ \varphi^{(i_2)} \circ \cdots \circ \varphi^{(i_l)}(\pi) \tag{4}$$

with $\varphi^{(i_k)} = \varphi_2^{(i_k)} \circ \varphi_1^{(i_k)}$ for $k = 1, \ldots, l$.

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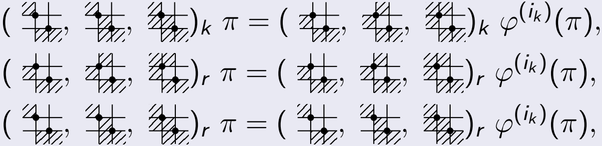
with $\varphi^{(i_k)} = \varphi_2^{(i_k)} \circ \varphi_1^{(i_k)}$ for $k = 1, \ldots, l$.

Lemma

The mapping $\varphi^{(i_k)}$ is an bijection such that for $\pi \in S_n$ and $r \neq k$,



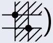
(5a)
(5b)
(5c)

where $(pattern)_k$ means the number of the patterns between $\pi(i_{k-1})$ and $\pi(i_k)$.

Main results II

Theorem (Han-Zeng)

There exist an involution Ψ on S_n such that for $\pi \in S_n$,

(, , )$(\pi) =$ (, , )$\Psi(\pi))$.

Further research

	Nr.	Repr. p	Ref.	Nr.	Repr. p	Ref.
proved equidistributions	48		[12, Theorem 5.1]			
	49					
conjectured equidistributions	23		Theorem 1.9	53		Theorem 1.6
	24			54		
	48		Theorem 1.6 and [12, Theorem 5.1]	57		N/A
	49			58		
	50			61		N/A
				62		

TABLE 1. Equidistributions for which enumeration is unknown. Pattern's numbers are adopted from [6, 12]

Further research

- A direction is studying joint distribution of patterns considered in this paper and other permutation statistics, for example, descent and inversion.
- it would be interesting to classify completely mesh patterns of length 2 with respect to their distribution.

B. Han and J. Zeng, Equidistributions of mesh patterns of length two, Sém. Lothar. Combin. 85B (2021), Art. 12, 12 pp.

B. Han and J. Zeng, Equidistributions of mesh patterns of length two and Kitaev and Zhang's conjectures, Adv. in Appl. Math. 127 (2021), Paper No. 102149, 17 pp.

Thank You !