Combinatorial Howe duality of symplectic type

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0. Introduction

Motivation (Howe duality of symplectic type)
For some choices of $\mathbb{Z}_2$-graded sets $\mathcal{A}$, there exist

- a Lie (super) algebra $\mathfrak{g}_{\mathcal{A}}$ and
- a semisimple $\mathfrak{g}_{\mathcal{A}}$-module $\mathscr{F}_{\mathcal{A}}$ (Fock space)

such that $\exists$ $(\mathfrak{g}_{\mathcal{A}}, \mathrm{Sp}_{2\ell})$-bimodule decomposition (1) with

$$\mathscr{F}_{\mathcal{A}}^{\otimes \ell} \cong \bigoplus_{\lambda \in \mathscr{P}_\ell} V_{\mathfrak{g}_{\mathcal{A}}}(\lambda, \ell) \otimes V_{\mathrm{Sp}_{2\ell}}(\lambda) \quad (1)$$

- $V_{\mathfrak{g}_{\mathcal{A}}}(\lambda, \ell)$: irreducible $\mathfrak{g}_{\mathcal{A}}$-module
- $V_{\mathrm{Sp}_{2\ell}}(\lambda)$: irreducible $\mathrm{Sp}_{2\ell}$-module

Example · $\mathfrak{g}_{\mathcal{A}} = \mathfrak{sp}_{2n}$ for $\mathcal{A} = [\overline{n}]$ [Howe 95] · $\mathfrak{g}_{\mathcal{A}} = \mathfrak{osp}_{2m|2n}$ for $\mathcal{A} = \mathbb{I}_{m|n}$ [Cheng-Zhang 04]

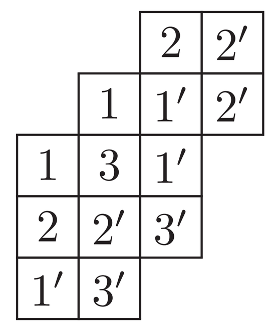
Goal Construct a symplectic analogue of RSK algorithm describing (1) for arbitrary $\mathcal{A}$

1. Notations

- $\mathcal{A}$: $\mathbb{Z}_2$-graded ordered set (letter set)
  e.g. $[\overline{n}] = \{\overline{n} < \cdots < \overline{1}\}$ with all even degree
  $\mathbb{I}_{m|n} = \{\underbrace{1 < \cdots < m}_{even} < \underbrace{1' < \cdots < n'}_{odd}\}$
- $SST_{\mathcal{A}}(\lambda/\mu)$: the set of $\mathcal{A}$-semistandard tableaux of shape λ/μ

Example

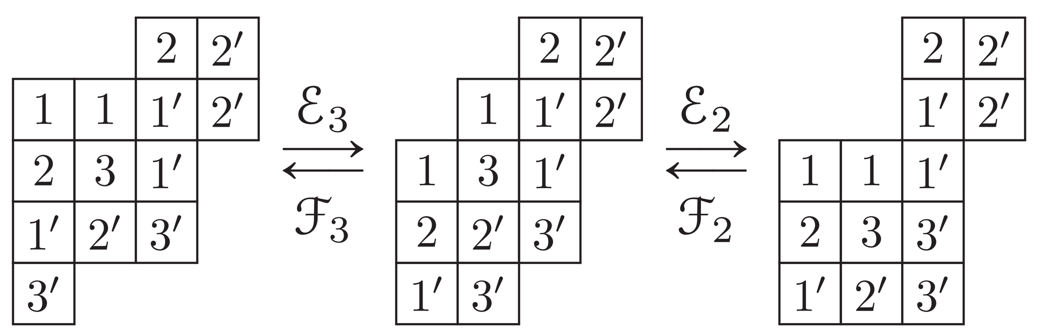
- $\mathcal{A} = \mathbb{I}_{4|3}$
- $\lambda = (4,4,3,3,2)$
- $\mu = (2,1)$


$\in SST_{\mathcal{A}}(\lambda/\mu)$

2. Combinatorial model of $\mathscr{F}_{\mathcal{A}}^{\otimes \ell}$

- $\mathbf{E}_{\mathcal{A}}^{2\ell} = \bigsqcup SST_{\mathcal{A}}((1^{u_{2\ell}})) \times \cdots \times SST_{\mathcal{A}}((1^{u_1}))$
  : the set of (2ℓ)-tuples of column tableaux
- $\mathbf{E}_{\mathcal{A}}^{2\ell}$ is a regular $\mathfrak{gl}_{2\ell}$-crystal with crystal operators $\mathcal{E}_i$ and $\mathcal{F}_i$ $(1 \leqslant i \leqslant 2\ell - 1)$, which acts on columns by jeu de taquin slides.

Example (on $\mathbf{E}_{\mathcal{A}}^4$ with $\mathcal{A} = \mathbb{I}_{4|3}$)



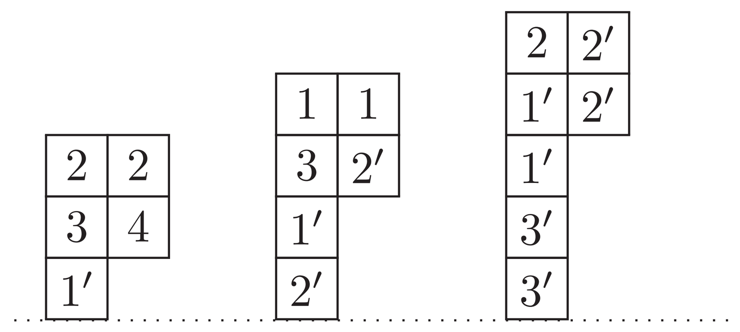
3. Spinor model $\mathbf{T}_{\mathcal{A}}(\lambda, \ell)$

- $\mathbf{T}_{\mathcal{A}}(a) = \bigsqcup_{n \geqslant 0} SST_{\mathcal{A}}((2^n, 1^a)) \subseteq \mathbf{E}_{\mathcal{A}}^2$ $(a \geqslant 0)$
- $\mathbf{T}_{\mathcal{A}}(a_\ell, \ldots, a_1) = \mathbf{T}_{\mathcal{A}}(a_\ell) \times \cdots \times \mathbf{T}_{\mathcal{A}}(a_1)$

Spinor model [Kwon 15] $\lambda \in \mathscr{P}_\ell$

- $\mathbf{T}_{\mathcal{A}}(\lambda, \ell)$: subset of $\mathbf{T}_{\mathcal{A}}(\lambda_\ell, \ldots, \lambda_1)$ satisfying the *admissibility conditions*
- $\exists$ bijection between $\mathbf{T}_{[\overline{n}]}(\lambda, \ell)$ and the set of KN tableaux [Kashiwara-Nakashima 94] of shape $\rho_n(\lambda, \ell)$

Example ($\mathcal{A} = \mathbb{I}_{4|3}$)



$(T_3, T_2, T_1) \in \mathbf{T}_{\mathcal{A}}((3,2,1),3)$

Character of $\mathbf{T}_{\mathcal{A}}(\lambda, \ell)$

$$S_{(\lambda,\ell)}(\mathbf{x}_{\mathcal{A}}) = \sum_{T \in \mathbf{T}_{\mathcal{A}}(\lambda,\ell)} \mathbf{x}_{\mathcal{A}}^T$$

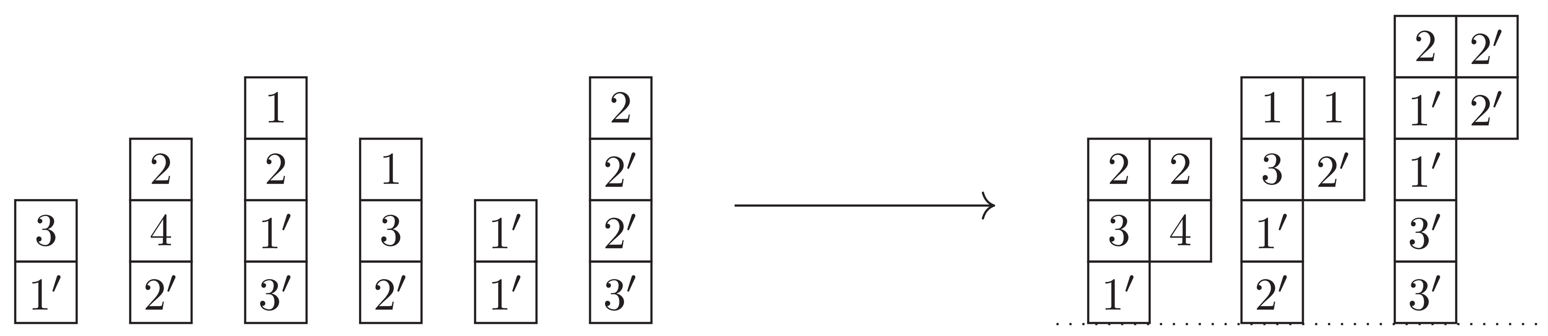
- $\mathbf{x}_{\mathcal{A}}^T = \prod x_a^{m_a}$ with $m_a = \#$(occurrences of $a \in \mathcal{A}$ in T)
- $S_{(\lambda,\ell)}(\mathbf{x}_{\mathcal{A}}) = \mathrm{ch} V_{\mathfrak{g}_{\mathcal{A}}}(\lambda, \ell)$ (see (1))

4. Insertion tableaux

For $T \in \mathbf{E}_{\mathcal{A}}^{2\ell}$, we construct
(1) a combinatorial algorithm, called **a symplectic jeu de taquin for spinor models**, and
(2) a tableau $\mathtt{P}(T) \in \mathbf{T}_{\mathcal{A}}(\lambda, \ell)$ for some $\lambda \in \mathscr{P}_\ell$.

- $\mathtt{P}(T)$ is obtained by applying a sequence of $\mathcal{E}_i$ and $\mathcal{F}_i$.
- Our algorithm for $\mathcal{A} = [\overline{n}]$ is compatible with a symplectic jeu de taquin for KN tableaux [Sheats 99].

Example ($\mathcal{A} = \mathbb{I}_{4|3}$) $\mathtt{P}(T) = \mathcal{E}_4 \mathcal{F}_3 \mathcal{F}_4 \mathcal{E}_2 \mathcal{E}_3 \mathcal{F}_1^2 \mathcal{F}_5 T$



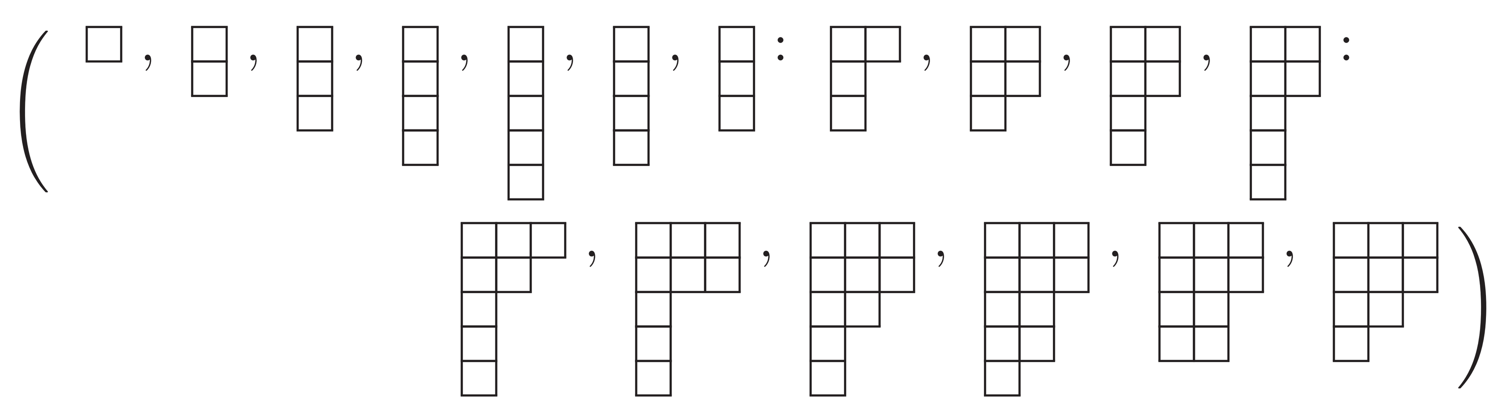
$T \in \mathbf{E}_{\mathcal{A}}^6$ $\longrightarrow$ $\mathtt{P}(T) \in \mathbf{T}_{\mathcal{A}}((3,2,1),3)$

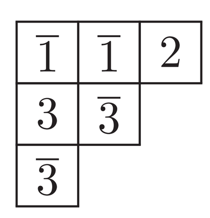
5. Recording tableaux

- An oscillating tableau is a sequence $(Q_1, \ldots, Q_\ell)$ of partitions with $Q_i/Q_{i+1} = \square$ or $Q_{i+1}/Q_i = \square$ for all i
- $\mathbf{K}(\lambda, \ell)$: the set of King tableaux of shape $\lambda \in \mathscr{P}_\ell$ [King 75]
- [Lee 19]
  $\exists$ bijection between $\mathbf{K}(\lambda, \ell)$ and a set of oscillating tableaux $\mathbf{O}(\lambda, \ell)$

For $T \in \mathbf{E}_{\mathcal{A}}^{2\ell}$, we define
an oscillating tableau $Q(T) \in \mathbf{O}(\lambda, \ell) \overset{1-1}{\longleftrightarrow}$ a King tableau $\mathtt{Q}(T) \in \mathbf{K}(\lambda, \ell)$.

Example (continued)

$Q(T) =$


$\mathtt{Q}(T) =$


6. Symplectic RSK correspondence

Theorem (Heo-Kwon 20). *For $\ell \geqslant 1$, we have a bijection*

$$\mathbf{E}_{\mathcal{A}}^{2\ell} \longrightarrow \bigsqcup_{\lambda \in \mathscr{P}_\ell} \mathbf{T}_{\mathcal{A}}(\lambda, \ell) \times \mathbf{K}(\lambda, \ell), \qquad T \longmapsto (\mathtt{P}(T), \mathtt{Q}(T))$$

Corollary (Cauchy-type identity). *We have the following identity.*

$$t^\ell \prod_{j=1}^{\ell} \frac{\prod_{a \in \mathcal{A}_0}(1 + x_a z_j)(1 + x_a z_j^{-1})}{\prod_{a \in \mathcal{A}_1}(1 - x_a z_j)(1 - x_a z_j^{-1})} = \sum_{\lambda \in \mathscr{P}_\ell} S_{(\lambda,\ell)}(\mathbf{x}_{\mathcal{A}}) sp_\lambda(\mathbf{z}).$$

- When $\mathcal{A} = \mathcal{A}_0$ and $\mathcal{A} = \mathcal{A}_1$, we recover the identities obtained by King and Littlewood-Weyl, respectively.

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