Bi-symmetric multiple equidistributions on ascent sequences

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Outline:

1. The Fishburn family and Eulerian/Stirling statistics
2. Generating functions and equidistributions
3. Transformations of basic hypergeometric series
4. Connections to permutations and matrices
5. Some open questions

1. The Fishburn family and Eulerian/Stirling statistics

Fishburn numbers

Fishburn numbers are the coefficients of the formal power series

$$\sum_{m=0}^{\infty}\prod_{i=1}^{m}(1-(1-z)^i) = 1 + z + 2z^2 + 5z^3 + 15z^4 + 53z^5 + \cdots.$$

This generating function was derived by Zagier (2001). Subsequently Andrews and Jelínek (2013) found an equivalent form:

$$\sum_{m=0}^{\infty}\prod_{i=1}^{m}(1-(1-z)^i) = \sum_{k=0}^{\infty}\frac{1}{(1-z)^{k+1}}\prod_{i=1}^{k}\left(1-\left(\frac{1}{1-z}\right)^i\right)^2.$$

by applying the Rogers–Fine identity:

$$\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(bq;q)_n}t^n = \sum_{n=0}^{\infty}\frac{(aq;q)_n(\frac{atq}{b};q)_n b^n t^n q^{n^2}(1-atq^{2n+1})}{(bq;q)_n(t;q)_{n+1}}$$

holds when $|q|<1$, $|t|<1$ and $b \neq q^k$ for $k<0$.

Fishburn numbers

The study of Fishburn numbers and their generalizations has remarkably led to many interesting results, including for instance

- Congruences (Garvan 2015, Andrews–Sellers 2016, Bijaoui–Boden–Myers–Osburn–Rushworth–Tronsgard–Zhou 2020),
- Asymptotic formulas (Zagier 2001, Jelínek 2012, Bringmann–Li–Rhoades 2014, Hwang–J. 2019),
- q-series (Andrews–Jelínek 2013, J.–Schlosser 2020),
- A variety of bijections (Bousquet-Mélou–Claesson–Dukes–Kitaev 2010, Claesson–Linusson 2011, Dukes–Parviainen 2010, Levande 2013, Fu–J.–Lin–Yan–Zhou 2019, Dukes–McNamara 2019, Auli–Elizalde 2020),
- Generating functions (Bousquet-Mélou–Claesson–Dukes–Kitaev 2010, Dukes–Kitaev–Remmel–Steingrímsson 2011, Jelínek 2012, Zagier 2001).

Fishburn family

Fishburn matrices
dimension
1st row sum
wNE-cells

$(\mathbf{2}–\mathbf{1})$-avoiding inversion seqs.
rep
max

Ascent sequences
asc, rep
zero, max
rmin

$(2|\bar{3}1)$-avoiding perms.
des, iasc
lmin, lmax, rmax

$(\mathbf{2}+\mathbf{2})$-free posets
magnitude
minimal elements

Regular linearized chord diagrams
length of the initial run of openers
pairs of arcs $(a, b), (c, d)$ such that $a < b = c - 1 < d - 1$

Blue: Eulerian statistics
Red: Stirling statistics

Ascent sequences

An inversion sequence $s = (s_1, \ldots, s_n)$ is a sequence of non-negative integers such that $0 \leq s_i \leq i - 1$. An ascent sequence is an inversion sequence with $s_1 = 0$ and

$$0 \leq s_i \leq \mathrm{asc}(s_1, \ldots, s_{i-1}) + 1.$$

For example, $(0, 1, 2, 0, 3, 2, 1, 5)$ is not an ascent sequence, but $(0, 1, 2, 0, 3, 2, 1, 4)$ is an ascent sequence.

ascent: $\mathrm{asc}(s) = |\{i \in [n-1] : s_i < s_{i+1}\}|$,

repeat: $\mathrm{rep}(s) = n - |\{s_1, s_2, \ldots, s_n\}|$,

maximal: $\max(s) = |\{i \in [n] : s_i = i - 1\}|$,

zero: $\mathrm{zero}(s) = |\{i \in [n] : s_i = 0\}|$.

rmin: $\mathrm{rmin}(s) = |\{i \in [n] : s_i < s_j \text{ for all } j > i\}|$.

For $s = (0, 1, 2, 0, 3, 2, 1, 4)$, $\mathrm{asc}(s) = 4$, $\mathrm{rep}(s) = 3$, $\mathrm{rmin}(s) = 3$, $\max(s) = 3$, $\mathrm{zero}(s) = 2$.

2. Generating functions and equidistributions

A refined generating function

$G(t; x, q, u, z)$ is the generating function of ascent sequences counted by the length (variable t), rep (variable x), max (variable q), asc (variable u) and zero (variable z):

$$G(t; x, q, u, z) = \sum_{n \geq 1} \left(\sum_{s \in \mathcal{A}_n} x^{\mathrm{rep}(s)} q^{\mathrm{max}(s)} u^{\mathrm{asc}(s)} z^{\mathrm{zero}(s)} \right) t^n .$$

Theorem (Fu-J.-Lin-Yan-Zhou, 2018): The generating function

$$G(t; x, q, u, z) = \sum_{m \geq 0} \frac{zqrx^m (1 - qr)(1 - r)^m (x + u - xu)}{[x(1 - u) + u(1 - qr)(1 - r)^m][x + u(1 - x)(1 - qr)(1 - r)^m]} \times \prod_{i=0}^{m-1} \frac{1 + (zr - 1)(1 - qr)(1 - r)^i}{x + u(1 - x)(1 - qr)(1 - r)^i},$$

where $r = t\,(x + u - xu)$.

Consequences of the refined generating function

1. The pair (zero, max) is symmetric on ascent sequences.

$$G(t;1,q,1,z)=\sum_{m\geq 0} qzt\prod_{i=0}^{m-1}[1-(1-zt)(1-qt)(1-t)^i].$$

   It affirms the conjecture by Kitaev, Remmel, Dukes and Parviainen.

2. Together with the following theorem,
   Theorem (Fu-J.-Lin-Yan-Zhou, 2018): There is a bijection $\Upsilon:\mathcal{A}_n\to\mathcal{A}_n$ which transforms the quadruple (asc, rep, zero, max) to (rep, asc, rmin, zero).

$$\implies G(t;1,1,u,z)=\sum_{m\geq 0} u^m\prod_{i=0}^{m}\frac{1-(1-zt)(1-t)^i}{u+(1-u)(1-zt)(1-t)^i},$$

$$=\sum_{m\geq 0}\frac{zt(1-t)^{m+1}}{1-u+u(1-t)^{m+1}}\prod_{i=0}^{m-1}(1-(1-zt)(1-t)^{i+1}).$$

   we are able the find two equivalent forms for $G(t;1,1,u,z)$. The first one was also derived by Jelínek, while the second one appears to be new.

Two remaining questions in 2018

- Conjecture (Fu-J.-Lin-Yan-Zhou, 2018):
  There is a bijection $\Phi : \mathcal{A}_n \to \mathcal{A}_n$ such that for all $s \in \mathcal{A}_n$,
  $$(\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{max})s = (\mathsf{rep}, \mathsf{asc}, \mathsf{max}, \mathsf{zero})\Phi(s).$$
  In terms of generating functions, it is equivalent to prove that
  $$G(t; x, q, u, z) = G(t; u, z, x, q).$$
- Can we prove these equivalent forms directly?
  $$\sum_{m\geq 0} u^m \prod_{i=0}^{m} \frac{1-(1-zt)(1-t)^i}{u+(1-u)(1-zt)(1-t)^i},$$
  $$= \sum_{m\geq 0} \frac{zt(1-t)^{m+1}}{1-u+u(1-t)^{m+1}} \prod_{i=0}^{m-1} (1-(1-zt)(1-t)^{i+1}).$$

The second part of this talk will answer these two questions.

Symmetric multiple equidistributions

Theorem (J.-Schlosser, 2020): There is a bijection $\Phi : \mathcal{A}_n \to \mathcal{A}_n$ such that for all $s \in \mathcal{A}_n$,

$$(\mathrm{asc}, \mathrm{rep}, \mathrm{zero}, \mathrm{max}, \mathrm{rmin}, \mathrm{ealm}, \mathrm{rpos})s$$
$$= (\mathrm{asc}, \mathrm{rep}, \mathrm{zero}, \mathrm{rmin}, \mathrm{max}, \mathrm{rpos}, \mathrm{ealm})\Phi(s).$$

Two new auxiliary statistics: ealm and rpos

ealm: the entry after the last maximal entry.

For instance, $\mathrm{ealm}(0, 1, 2, \mathbf{0}, 3, 2, 1, 4) = 0$.

When $\max(s) = |s|$, we set $\mathrm{ealm}(s) = 0$.

rpos: the maximal m appearing as an m-th right-to-left minimum (rmin) and that appearing at least twice after the $(m-1)$-th rmin. If no such m exists or $\mathrm{rmin}(s) = |s|$, set $\mathrm{rpos}(s) = 0$.

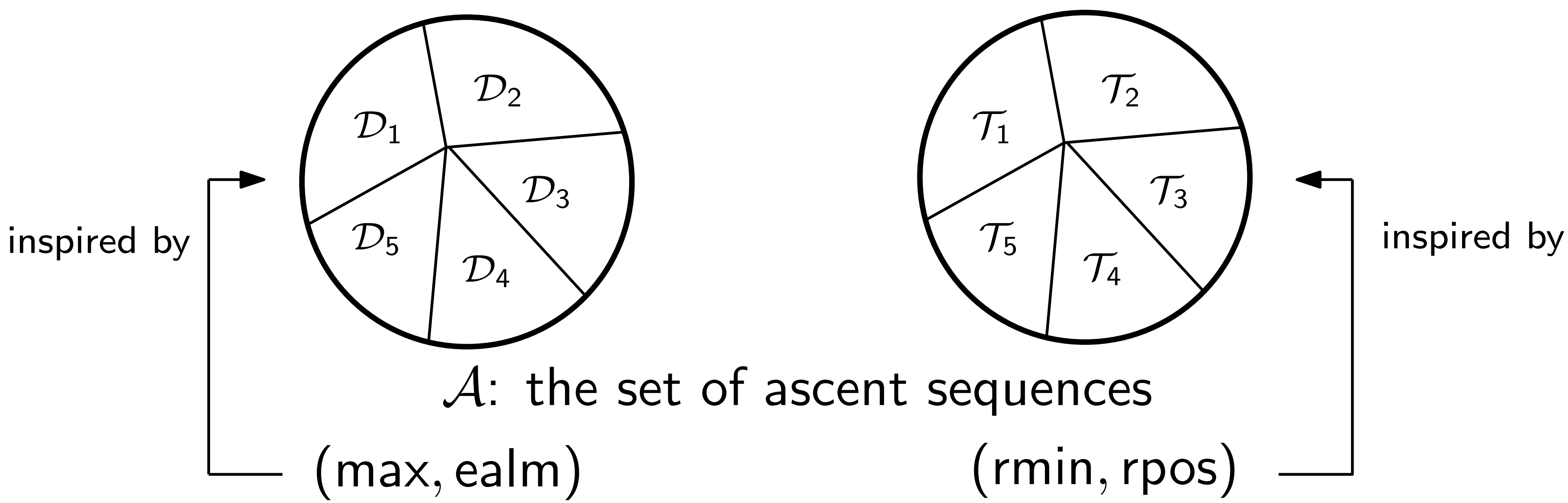
For instance, $\mathrm{rpos}(\mathbf{0}, 1, 2, 0, 3, 2, 1, 4) = 0$.

$\mathrm{rpos}(0, 1, 2, 0, \mathbf{1}, 3, 2, 1, 4) = 1$.

Main idea of the bijection Φ

$$(\mathrm{asc}, \mathrm{rep}, \mathrm{zero}, \mathrm{max}, \mathrm{rmin}, \mathrm{ealm}, \mathrm{rpos})s$$
$$= (\mathrm{asc}, \mathrm{rep}, \mathrm{zero}, \mathrm{rmin}, \mathrm{max}, \mathrm{rpos}, \mathrm{ealm})\Phi(s).$$

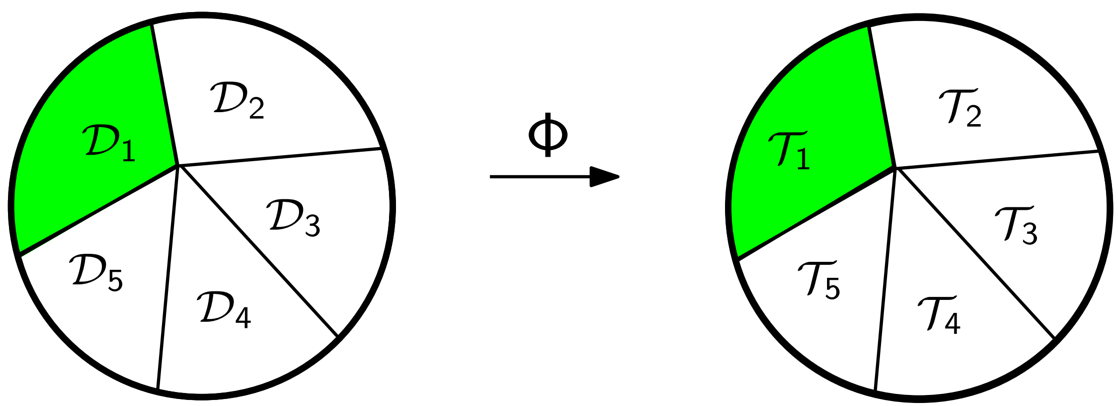
We take the Divide-and-Conquer strategy, namely we find two parallel decompositions of ascent sequences and establish the bijection $\Phi : \mathcal{D}_i \to \mathcal{T}_i$ for each subset.



(Fu-J.-Lin-Yan-Zhou, 2018) (J.-Schlosser, 2020)

Main idea of the bijection Φ

$$(\mathrm{asc}, \mathrm{rep}, \mathrm{zero}, \max, \mathrm{rmin}, \mathrm{ealm}, \mathrm{rpos})s$$
$$= (\mathrm{asc}, \mathrm{rep}, \mathrm{zero}, \mathrm{rmin}, \max, \mathrm{rpos}, \mathrm{ealm})\Phi(s).$$



$\Phi : \mathcal{D}_1 \cap \mathcal{A}_n \to \mathcal{T}_1 \cap \mathcal{A}_n$, that is,

the bijection $\Phi : \{s : |s| - \max(s) = 1\} \to \{s : |s| - \mathrm{rmin}(s) = 1\}$ is explicitly defined, which forms an inductive basis to construct Φ for other subsets with larger value of $|s| - \max(s)$ or $|s| - \mathrm{rmin}(s)$.

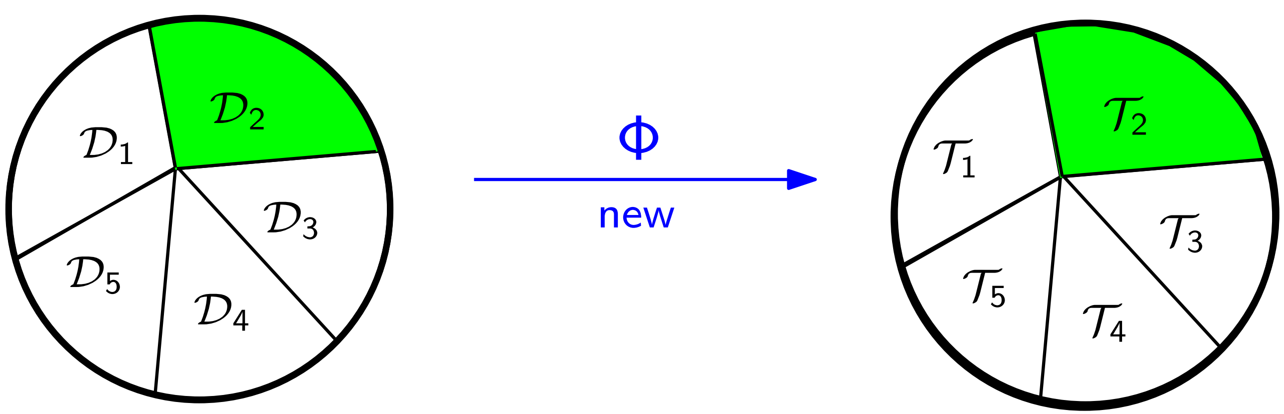
Main idea of the bijection Φ

$$(\text{asc}, \text{rep}, \text{zero}, \max, \text{rmin}, \text{ealm}, \text{rpos})s = (\text{asc}, \text{rep}, \text{zero}, \text{rmin}, \max, \text{rpos}, \text{ealm})\Phi(s).$$

with smaller $|s| - \max(s)$ — with smaller $|s| - \text{rmin}(s)$

$$\{(i, s) : s \in \mathcal{A}_{n-1}, |s| \neq \max(s), \text{ealm}(s) \leq i < \max(s)\} \xrightarrow{\Phi} \{(i, s) : s \in \mathcal{A}_{n-1}, |s| \neq \text{rmin}(s), \text{rpos}(s) \leq i < \text{rmin}(s)\}$$

1-1 (is known by induction) 1-1



This idea is also applied to construct Φ for $\mathcal{D}_3$ to $\mathcal{D}_5$.

Application of the bijection Φ

$\mathcal{G}(t;x,q,u,v)$ is the generating function of ascent sequences counted by the length (variable t), rep (variable x), max (variable q), asc (variable u) and rmin (variable v):

$$\mathcal{G}(t;x,q,u,v) = \sum_{n\geq 1}\left(\sum_{s\in\mathcal{A}_n} x^{\mathsf{rep}(s)} q^{\mathsf{max}(s)} u^{\mathsf{asc}(s)} v^{\mathsf{rmin}(s)}\right) t^n.$$

Theorem (J.-Schlosser, 2020): The generating function

$$\mathcal{G}(t;x,q,u,v) = \frac{vqt}{1-vqtu} + \sum_{m=0}^{\infty} \frac{rv(1-qr)(1-r)^m}{(x-xu+u(1-qr)(1-r)^m)(1-qtuv)}$$

$$\times \prod_{i=0}^{m} \frac{x(1-(1-qr)(1-r)^i)(x-xu+u(1-qr)(1-r)^i)}{(x-u(x-1)(1-qr)(1-r)^i)(x-xu+u(1-rv)(1-qr)(1-r)^i)}$$

where $r = t\,(x+u-xu)$.

3. Transformations of basic hypergeometric series

Transformations of basic hypergeometric series

For indeterminates a and q (the latter is referred to as the base), and non-negative integer k, the basic shifted factorial (or q-shifted factorial) is defined as

$$(a;q)_k := \prod_{j=1}^{k}(1-aq^{j-1}), \text{ also for } k=\infty.$$

For brevity, we write

$$(a_1,\ldots,a_m;q)_k := (a_1;q)_k\cdots(a_m;q)_k.$$

${}_\alpha\phi_\beta$ basic hypergeometric series:

$${}_\alpha\phi_\beta\left[\begin{matrix}a_1,\ldots,a_\alpha\\ b_1,\ldots,b_\beta\end{matrix};q,z\right] := \sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_\alpha;q)_k z^k}{(q,b_1,\ldots,b_\beta;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+\beta-\alpha}$$

where lower parameters b_i's are assumed to be chosen such that no poles occur in the summands of the series.

Transformations of basic hypergeometric series

One of the most important identities in the theory of basic hypergeometric series is the Sears transformation (1951):

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, c \\ d, e, abcq^{1-n}/de \end{matrix}; q, q\right]
$$

$$
= \frac{(e/a, de/bc; q)_n}{(e, de/abc; q)_n}\ {}_4\phi_3\left[\begin{matrix} q^{-n}, a, d/b, d/c \\ d, aq^{1-n}/e, de/bc \end{matrix}; q, q\right]
$$

For non-terminating basic hypergeometric series in base q we usually consider expansions around $q = 0$.

When dealing with generating functions for objects of the Fishburn family, we are treating power series in $r = t(x + u - xu)$, expanded around $r = 0$, that can be written as basic hypergeometric series in base $q = 1 - r$, thus can be viewed as functions analytic around $q = 1$.

Transformations of basic hypergeometric series

Theorem (J.-Schlosser, 2020): Let a, b, c, d, e, r be complex variables, j be a non-negative integer. Then, assuming that none of the denominator factors on both sides of the identity have vanishing constant term in r, we have the following transformation of convergent power series in a and r:

$$
{}_4\phi_3\left[\begin{matrix} (1-r)^j, 1-a, b, c \\ d, e, (1-r)^{j+1}(1-a)bc/de \end{matrix}; 1-r, 1-r\right]
$$

$$
= \frac{((1-r)e, (1-r)(1-a)bc/de; 1-r)_j}{((1-r)(1-a)/e, (1-r)bc/de; 1-r)_j}
$$

$$
\times {}_4\phi_3\left[\begin{matrix} (1-r)^j, 1-a, d/b, d/c \\ d, de/bc, (1-r)^{j+1}(1-a)/e \end{matrix}; 1-r, 1-r\right].
$$

Transformations of basic hypergeometric series

By letting $a \to 1$ in the new ${}_4\phi_3$ transformation we obtain the following result:

Corollary 1: Let b, c, d, e, r be complex variables, j be a non-negative integer. Then, assuming that none of the denominators on both sides of the identity have vanishing constant term in r, we have the following transformation of convergent power series in r:

$$
{}_3\phi_2\left[\begin{matrix}(1-r)^j, b, c \\ d, e\end{matrix}; 1-r, 1-r\right]
= \frac{((1-r)/e; 1-r)_j}{((1-r)bc/de; 1-r)_j}\, {}_3\phi_2\left[\begin{matrix}(1-r)^j, d/b, d/c \\ d, de/bc\end{matrix}; 1-r, 1-r\right].
$$

$\Longrightarrow \mathcal{G}(t; x, q, u, v) = \mathcal{G}(t; x, v, u, q)$. That is,

$$(\mathrm{asc}, \mathrm{rep}, \mathrm{max}, \mathrm{rmin}) \sim (\mathrm{asc}, \mathrm{rep}, \mathrm{rmin}, \mathrm{max}).$$

Transformations of basic hypergeometric series

Corollary 2: Let a, b, c, e, r be complex variables, j be a non-negative integer. Then, assuming that none of the denominators on both sides of the identity have vanishing constant term in r, we have the following transformation of convergent power series in a and r:

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix}(1-r)^j, 1-a, b\\ e, (1-r)^{j+1}(1-a)b/ce\end{matrix}; 1-r, 1-r\right]\\
&= \frac{((1-r)/e; (1-r)(1-a)b/ce; 1-r)_j}{((1-r)(1-a)/e; (1-r)b/ce; 1-r)_j}\\
&\qquad \times {}_3\phi_2\left[\begin{matrix}(1-r)^j, 1-a, c\\ ce/b, (1-r)^{j+1}(1-a)/e\end{matrix}; 1-r, 1-r\right].
\end{aligned}
$$

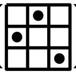
$\Longrightarrow G(t; x, q, u, z) = G(t; u, z, x, q)$. That is,

(asc, rep, max, zero) $\sim$ (rep, asc, zero, max).

4. Connections to permutations and matrices

Connections to permutations and matrices

All aforementioned (bi)-symmetric distributions on ascent sequences have counterparts over other members of the Fishburn family.

Corollary: There are three bijections between $S_n($$)$ and itself such that the following three (bi)-symmetric equidistributions hold, respectively:

$$(\mathrm{des}, \mathrm{iasc}, \mathrm{lmax}, \mathrm{lmin}, \mathrm{rmax}) \sim (\mathrm{des}, \mathrm{iasc}, \mathrm{lmax}, \mathrm{rmax}, \mathrm{lmin}),$$

$$(\mathrm{des}, \mathrm{iasc}, \mathrm{lmax}, \mathrm{lmin}) \sim (\mathrm{iasc}, \mathrm{des}, \mathrm{lmin}, \mathrm{lmax}),$$

$$(\mathrm{des}, \mathrm{iasc}, \mathrm{lmax}, \mathrm{rmax}) \sim (\mathrm{iasc}, \mathrm{des}, \mathrm{rmax}, \mathrm{lmax}).$$

Proof ingredients:

the bijection Φ + a bijection by Fu-J.-Lin-Yan-Zhou 2018
(which applies the bijection by
Baril-Vajnovszki, 2017)

Connections to permutations and matrices

Corollary: There is a bijection between $\mathcal{F}_n$ and itself such that the following symmetric distribution holds:

$$(\mathrm{rowsum}_1, \mathrm{ne}, \mathrm{mtr}) \sim (\mathrm{rowsum}_1, \mathrm{mtr}, \mathrm{ne}).$$

$\mathcal{F}_n$: the set of Fishburn matrices of size (the sum of entries) n.

$\mathrm{rowsum}_1(M) :=$ the sum of entries in the first row of M.

$\mathrm{ne}(M) :=$ the number of weakly north-east cells of M.

$\mathrm{mtr}(M) :=$ the largest index i such that the submatrix $(M_{s,t})_{s\leq i-1, t\leq i-1}$ is empty or is an identity matrix.

Proof ingredients:

the bijection Φ + a bijection by Chen-Yan-Zhou 2019 (which partially solves a conjecture by Jelínek 2015.)

5. Some open questions

Some open questions

1. Can we find a unified generating function of ascent sequences with respect to the statistics asc, rep, zero, max and rmin?

2. Do other members of the Fishburn family (other than ascent sequences and pattern-avoiding permutations) have these (bi)-symmetric distributions?

Fishburn matrices
dimension
1st row sum
wNE-cells

(**2–1**)-avoiding inversion seqs.
rep
max

(**2+2**)-free posets
magnitude
minimal elements

Regular linearized chord diagrams
length of the initial run of openers
pairs of arcs $(a, b), (c, d)$ such that $a < b = c - 1 < d - 1$

Blue: Eulerian statistics
Red: Stirling statistics

Some open questions

3. Is there a connection between permutation statistics and representation theory? Can one prove the (bi)-symmetric distributions in the language of representation theory?

Thank you for your attention!