q-Whittaker functions, finite fields, and Jordan forms

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Schur functions and Cauchy identity

- e.g. $s_{(2,1)}(x_1, x_2, x_3) =$

$$
\begin{array}{ccccccccccccccc}
x_1^2x_2 & + & x_1^2x_3 & + & x_1x_2^2 & + & x_1x_2x_3 & + & x_1x_2x_3 & + & x_1x_3^2 & + & x_2^2x_3 & + & x_2x_3^2 \\
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & &
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & &
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & &
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & &
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & &
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & &
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & &
\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}
\end{array}
$$

Theorem (Cauchy)

$$\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y})$$

- The left-hand side counts *nonnegative-integer matrices*, and the right-hand side counts *pairs of semistandard tableaux of the same shape*.
- e.g. Taking the coefficient of $x_1x_2y_1y_2$ on each side gives

$$
\begin{array}{ccccccc}
1 & + & 1 & = & 1 & + & 1 \\
12 & & 21 & & \left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \right) & & \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \right)
\end{array}
$$

Burge correspondence (1974)

- The *Burge correspondence* (also known as *column RSK*) is a bijection

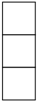
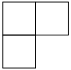
$$M \mapsto (\mathrm{P}(M), \mathrm{Q}(M))$$

between nonnegative-integer matrices and pairs of semistandard tableaux of the same shape. It proves the Cauchy identity for Schur functions.

- $\mathrm{P}(M)$ is obtained via *column insertion* and $\mathrm{Q}(M)$ via *recording*.
- e.g. $w = 25143$

$$\begin{array}{|c|}\hline 2 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline 5 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 5 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline \end{array} \qquad \underset{\mathrm{P}(w)}{\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}} \qquad\qquad \underset{\mathrm{Q}(w)}{\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}}$$

Nilpotent matrices

- Given an $n \times n$ nilpotent matrix N over $\Bbbk$, let $\mathrm{JF}^{\top}(N)$ be the *transposed Jordan form partition* of N.
- e.g. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto$  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mapsto$  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mapsto$ 

Theorem (Gansner (1981))

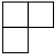
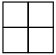
Let N be a generic $n \times n$ strictly upper-triangular matrix, where $N_{i,j} = 0$ for all inversions (i,j) of w^{-1}. Then $\mathrm{P}(w)$ and $\mathrm{Q}(w)$ can be read off from the Jordan forms of the leading submatrices of N and $w^{-1}Nw$.

- Gansner's result can be stated in terms of nilpotent operators acting on the *complete flag variety* over $\Bbbk$, discovered independently by Steinberg (1976, 1988) and Spaltenstein (1982).
- Rosso (2012) extended this result from permutations to all nonnegative-integer matrices, using the geometry of *partial flag varieties* over $\Bbbk$.

- e.g. $w = 25143$ $\qquad N = \begin{bmatrix} 0 & 0 & a & b & 0 \\ 0 & 0 & c & d & e \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\qquad (a, b, c, d, e \in \Bbbk \text{ generic})$

P(w):

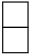
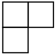
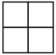
$$\begin{array}{cc} & 1 \\ 1 & \begin{bmatrix} 0 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 0 & a & b & 0 \\ 0 & 0 & c & d & e \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}$$

Q(w):

$$\begin{array}{cc} & 2 \\ 2 & \begin{bmatrix} 0 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{matrix} 2 & 5 \end{matrix} \\ \begin{matrix} 2 \\ 5 \end{matrix} & \begin{bmatrix} 0 & e \\ 0 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{matrix} 2 & 5 & 1 \end{matrix} \\ \begin{matrix} 2 \\ 5 \\ 1 \end{matrix} & \begin{bmatrix} 0 & e & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{matrix} 2 & 5 & 1 & 4 \end{matrix} \\ \begin{matrix} 2 \\ 5 \\ 1 \\ 4 \end{matrix} & \begin{bmatrix} 0 & e & 0 & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{matrix} 2 & 5 & 1 & 4 & 3 \end{matrix} \\ \begin{matrix} 2 \\ 5 \\ 1 \\ 4 \\ 3 \end{matrix} & \begin{bmatrix} 0 & e & 0 & d & c \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & a \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}$$

q-Whittaker functions and q-Cauchy identity

- The *q-Whittaker symmetric function* $W_\lambda(\mathbf{x};q)$ is a q-analogue of $s_\lambda(\mathbf{x})$.

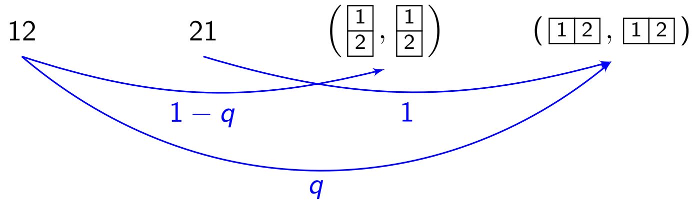
It is the $t=0$ specialization of the *Macdonald polynomial* $P_\lambda(\mathbf{x};q,t)$.

Theorem (Macdonald (1995))

$$\prod_{i,j\geq 1}\prod_{d\geq 0}\frac{1}{1-x_iy_jq^d} = \sum_\lambda \frac{(1-q)^{-\lambda_1}}{\prod_{i\geq 1}[\lambda_i-\lambda_{i+1}]_q!}W_\lambda(\mathbf{x};q)W_\lambda(\mathbf{y};q)$$

- e.g. Taking the coefficient of $x_1x_2y_1y_2$ on each side gives

$$(1-q)^{-2} \;+\; (1-q)^{-2} \;=\; (1-q)^{-1} \;+\; (1-q)^{-2}(1+q)$$



q-Burge correspondence

- e.g. $w = 12$ $\qquad N = \begin{bmatrix} 0 & \color{blue}{a} \\ 0 & 0 \end{bmatrix}$ $\qquad (a \in \mathbb{F}_{1/q})$

$$\mathrm{P}(w)\colon \quad \begin{array}{cc} & 1 \\ 1 & \begin{bmatrix} 0 \end{bmatrix} \end{array} \quad \begin{array}{ccc} & 1 & 2 \\ \begin{array}{c} 1 \\ 2 \end{array} & \multicolumn{2}{c}{\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}} \end{array} \qquad \mathrm{Q}(w)\colon \quad \begin{array}{cc} & 1 \\ 1 & \begin{bmatrix} 0 \end{bmatrix} \end{array} \quad \begin{array}{ccc} & 1 & 2 \\ \begin{array}{c} 1 \\ 2 \end{array} & \multicolumn{2}{c}{\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}} \end{array}$$

$$a \neq 0\colon \quad \begin{array}{|c|}\hline \\ \hline \end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline \\ \hline \end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \qquad \color{blue}{\mathbb{P} = 1 - q}$$

$$a = 0\colon \quad \begin{array}{|c|}\hline \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \qquad \color{blue}{\mathbb{P} = q}$$

- e.g. $w = 21$ $\qquad N = \begin{bmatrix} 0 & \color{red}{0} \\ 0 & 0 \end{bmatrix}$

$$\mathrm{P}(w)\colon \quad \begin{array}{cc} & 1 \\ 1 & \begin{bmatrix} 0 \end{bmatrix} \end{array} \quad \begin{array}{ccc} & 1 & 2 \\ \begin{array}{c} 1 \\ 2 \end{array} & \multicolumn{2}{c}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} \end{array} \qquad \mathrm{Q}(w)\colon \quad \begin{array}{cc} & 2 \\ 2 & \begin{bmatrix} 0 \end{bmatrix} \end{array} \quad \begin{array}{ccc} & 2 & 1 \\ \begin{array}{c} 2 \\ 1 \end{array} & \multicolumn{2}{c}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} \end{array}$$

$$\begin{array}{|c|}\hline \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \qquad \color{blue}{\mathbb{P} = 1}$$

q-Burge correspondence

- Let $1/q$ be a prime power, and fix partial flags $F \xrightarrow{M} F'$ over $\mathbb{F}_{1/q}$. Let N denote a uniformly random nilpotent matrix strictly compatible with both F and F'. For semistandard tableaux T and T' of the same shape, define

$$\mathrm{p}_M(T, T') := \mathbb{P}[\mathrm{JF}^\top(N; F) = T \text{ and } \mathrm{JF}^\top(N; F') = T'].$$

(This definition depends only on M, not on the choice of (F, F').)

Theorem (Karp, Thomas (2021+))

(i) The maps $\mathrm{p}_M(\cdot, \cdot)$ *define a probabilistic bijection proving the Cauchy identity for q-Whittaker functions, called the q-Burge correspondence.*

(ii) As q $\to 0$, *the q-Burge correspondence converges to the deterministic Burge correspondence.*

- It is an open problem to determine if $\mathrm{p}_M(T, T')$ is a polynomial in q.
- A different probabilistic bijection was given by Matveev and Petrov (2017), using the *q-row insertion* of Borodin and Petrov (2016).