Cluster Scattering Diagrams and Theta Functions for Reciprocal Generalized Cluster Algebras

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Preprint

Ordinary Cluster Algebras

Ordinary cluster algebras were defined by Fomin and Zelevinsky in 2001.

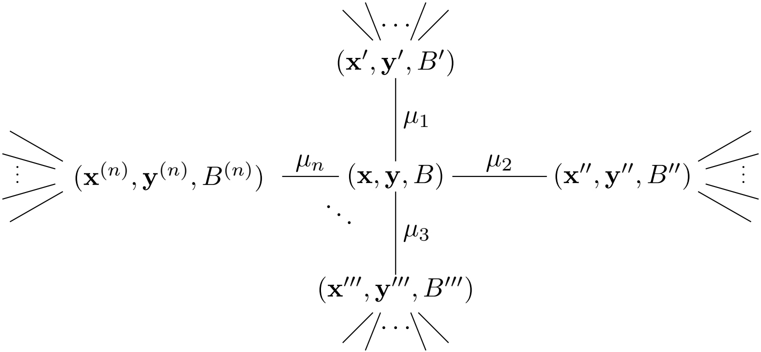
A cluster algebra is a type of commutative algebra whose generators, i.e. *cluster variables*, are related by standard *exchange relations*. The cluster variables appear in fixed size subsets called *clusters*, each of which suffices to generate the entire algebra. The clusters are related via *mutation*.

A cluster algebra is specified by a set of *seed data* $\Sigma = (\mathbf{x}, \mathbf{y}, B)$ where

- $\mathbf{x} = (x_1, \ldots, x_n)$ is the *initial cluster*,
- $\mathbf{y} = (y_1, \ldots, y_n)$ is the *initial coefficient cluster*,
- and the *exchange matrix* B is an $n \times n$ skew-symmetrizable matrix with entries in $\mathbb{Z}$.

Ordinary Cluster Algebras

We can picture the structure



where μ_k stands for *mutation in direction k*.

Ordinary Cluster Algebras

Algebraically, mutation in direction k is an involutive operation which maps a seed $\Sigma = (\mathbf{x}, \mathbf{y}, B)$ to another seed $\mu_k(\Sigma) = (\mathbf{x}', \mathbf{y}', B')$ via the relations:

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + [-b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+ & \text{else} \end{cases}$$

$$y'_i = \begin{cases} y_k^{-1} & \text{if } i = k; \\ y_i y_k^{[b_{ki}]_+} (1 \oplus y_k)^{-b_{ki}} & \text{if } i \neq k \end{cases}$$

$$x'_i = \begin{cases} x_i & \text{if } i \neq k \\ \frac{y_k \prod x_j^{[b_{ik}]_+} + \prod x_j^{[-b_{ik}]}}{x_k (1 \oplus y_k)} & \text{if } i = k \end{cases}$$

where $[a]_+ = \max(0, a)$.

Note that these exchange relations are always binomial.

Generalized Cluster Algebras

One natural way to generalize an ordinary cluster algebra, introduced by Chekhov and Shapiro in 2011, is to allow the exchange relations to be polynomials with arbitrarily many terms.

A *generalized cluster algebra* has seeds of the form $\Sigma = (\mathbf{x}, \mathbf{y}, B, \mathbf{r}, \mathbf{a})$ where

- $\mathbf{x}$ is the *initial cluster*,
- $\mathbf{y}$ is the *initial coefficient cluster*,
- B, the *exchange matrix*, is an $n \times n$ skew-symmetrizable matrix with integer entries,
- $\mathbf{r} = (r_1, \ldots, r_n)$ is an n-tuple of natural numbers with r_i specifying the degree of the i^{th} exchange polynomial,
- and $\mathbf{a} = (a_{i,s})_{i \in [n], s \in [r_i - 1]}$ is a collection of formal variables such that the coefficients of the i^{th} exchange polynomial are determined by the tuple $(a_{i,0}, a_{i,1}, \ldots, a_{i,r_i-1}, a_{i,r_i})$.

Generalized Cluster Algebras

Because the $a_{i,j}$ are formal variables, we must work over the ground ring $R := \mathbb{k}[a_{i,j}]$ where $\mathbb{k}$ is a field of characteristic zero, rather than over $\mathbb{k}$ as in the ordinary case.

We focus on a subtype of generalized cluster algebras called *reciprocal generalized cluster algebras*.

In these algebras, we require that

- $a_{i,1} = a_{i,r_i} = 1$
- and $a_{i,k} = a_{i,r_i-k}$ for all $2 \leq k \leq r_i - 1$.

Generalized Cluster Algebras

Here, mutation in direction k is defined by the exchange relations:

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + r_k\left([-b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+\right) & i,j \neq k \end{cases}$$

$$y'_i = \begin{cases} y_k^{-1} & i = k \\ y_i \left(y_k^{[b_{ki}]_+}\right)^{r_k} \left(1 \oplus a_{k,1} y_k \oplus \cdots \oplus a_{k,r_k-1} y_k^{r_k-1} \oplus y_k^{r_k}\right)^{-b_{ki}} & i \neq k \end{cases}$$

$$x'_i = \begin{cases} x_k^{-1}\left(\prod_{j=1}^n x_j^{[-b_{jk}]_+}\right)^{r_k} \frac{1 + a_{k,1}\widehat{y}_k + \cdots + \widehat{y}_k^{r_k}}{1 \oplus a_{k,1} y_k + \cdots \oplus a_{k,r_k-1} y_k^{r_k-1} \oplus y_k^{r_k}} & i = k \\ x_i & i \neq k \end{cases}$$

$$a'_{k,i} = a_{k,r_k-i}$$

where $\widehat{y}_k = y_k x_1^{b_{1k}} \cdots x_n^{b_{nk}}$.

Note: When all $r_i = 1$, we recover the ordinary exchange relations.

Scattering Diagrams

From a geometric point of view, we study cluster algebras by studying *cluster varieties*.

Cluster varieties appear in pairs $(\mathcal{A}, \mathcal{X})$ called *cluster ensembles*.

- The $\mathcal{A}$-variety encodes information about the cluster variables (x_i).
- The $\mathcal{X}$-variety encodes information about the coefficients (y_i).

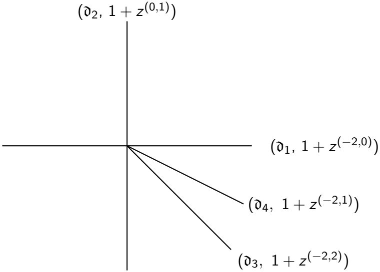
For the special case of cluster algebras with principal coefficients, we can consider the $\mathcal{A}$ *cluster variety with principal coefficients*, denoted $\mathcal{A}_{\text{prin}}$.

Scattering Diagrams

Scattering diagrams are a powerful tool from algebraic geometry which can be used to study the structure of cluster algebras.

Notably, Gross, Hacking, Keel, and Konstevich used cluster scattering diagrams to:

- Define theta functions and prove that they give a canonical positive basis for cluster algebras.
- Give a proof of positivity for arbitrary cluster algebras.



A scattering diagram is a collection of *walls* $\mathfrak{d}$ and automorphisms $f_{\mathfrak{d}}$.

Each wall is a codimension-1 cone (in rank 2, these are simply lines) and the automorphisms are formal power series in z.

Scattering Diagrams for Generalized Cluster Algebras

The following data is referred to as *generalized fixed data*, denoted Γ:

- The *cocharacter lattice* N with skew-symmetric bilinear form $\{\cdot,\cdot\} : N \times N \to \mathbb{Q}$.
- A saturated sublattice $N_{\mathrm{uf}} \subseteq N$ called the *unfrozen sublattice*.
- An index set I with $|I| = \mathrm{rank}(N)$ and subset $I_{\mathrm{uf}} \subseteq I$ such that $|I_{\mathrm{unf}}| = \mathrm{rank}(N_{\mathrm{uf}})$
- A set of positive integers $\{d_i\}_{i \in I}$ such that $\gcd(d_i) = 1$.
- A sublattice $N^\circ \subseteq N$ of finite index such that $\{N_{\mathrm{uf}}, N^\circ\} \subseteq \mathbb{Z}$ and $\{N, N_{\mathrm{uf}} \cap N^\circ\} \subseteq \mathbb{Z}$.
- A lattice $M = \mathrm{Hom}(N, \mathbb{Z})$ called the *character lattice* and sublattice $M^\circ = \mathrm{Hom}(N^\circ, \mathbb{Z})$.
- A set of positive integers $\{r_i\}_{i \in I}$.
- A collection $\{a_{i,j}\}_{i \in I_{\mathrm{uf}}, j \in [r_i - 1]}$ of formal variables.

Generalized Torus Seed Data

A *generalized torus seed* is a collection $\mathbf{s} = \{(e_i, \mathbf{a}_i)\}_{i \in I, s \in [r_i]}$ such that:

- $\{e_i\}_{i \in I}$ is a basis for N,
- $\{e_i\}_{i \in I_{\mathrm{uf}}}$ is a basis for N_{uf},
- $\{d_i e_i\}_{i \in I}$ is a basis for N°,
- $\{e_i^*\}_{i \in I}$ is a basis for the dual lattice M,
- $\{f_i = d_i^{-1} e_i^*\}_{i \in I}$ is a basis for M°,
- and each $(a_{i,j})$ is a tuple of formal variables taken from the collection specified in the generalized fixed data.

The generalized torus seed data defines a new bilinear form $[\cdot, \cdot]_{\mathbf{s}} : N \times N \to \mathbb{Q}$ given by $[e_i, e_j]_{\mathbf{s}} = \epsilon_{ij} = \{e_i, e_j\} d_j$.

Note: Placing the exchange polynomial coefficients $(a_{i,s})$ in the seed data rather than fixed data leaves open the possibility of extending this construction to non-reciprocal generalized cluster algebras.

Torus Seed Mutation

The generalized mutation relations for basis vectors e_i, f_i and the $\epsilon_{i,j}$ are:

$$e_i' := \begin{cases} e_i + r_k[\epsilon_{ik}]_+ e_k & i \neq k \\ -e_k & i = k \end{cases}$$

$$f_i' := \begin{cases} -f_k + r_k \sum_{j \in I_{\mathrm{uf}}} [-\epsilon_{kj}]_+ f_j & i = k \\ f_i & i \neq k \end{cases}$$

$$\epsilon_{ij}' = \begin{cases} -\epsilon_{ij} & k = i \text{ or } k = j \\ \epsilon_{ij} & k \neq i, j \text{ and } \epsilon_{ik}\epsilon_{kj} \leq 0 \\ \epsilon_{ij} + r_k|\epsilon_{ik}|\epsilon_{kj} & k \neq i, j \text{ and } \epsilon_{ik}\epsilon_{kj} \geq 0 \end{cases}$$

$$a_{k,s}' = a_{k, r_k - s}$$

Note: Torus seed mutation is only an involution up to isomorphism.

Some Translations

In the language of cluster algebras:

- The exchange matrix B encodes the ϵ_{ij} (i.e., the skew-symmetric form and choice of $\{d_i\}_{i\in I}$)
- The cluster variables are given by $x_i = z^{e_i}$ (the $\mathcal{A}$-variety)
- The coefficients are given by $y_i = z^{f_i}$ (the $\mathcal{X}$-variety)
- The index sets I and I_{uf} allow us to differentiate between frozen and unfrozen variables

The classic mutation relations can be derived from the mutation of the e_i, f_i, and ϵ_{ij} via simple algebra.

Example

Consider $\mathcal{A} = \left(\mathbf{x}, \mathbf{y}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, (3,1), ((1,a,a,1),(1,1))\right)$.

This generalized cluster algebra has fixed data:

- Skew-symmetric bilinear form given by the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
- Index sets $I = I_{\mathrm{uf}} = \{1,2\}$
- $d_1 = d_2 = 1$
- $r_1 = 3$ and $r_2 = 1$

Suppose we choose $\mathbf{s} = \{((1,0),(1,a,a,1)),((0,1),(1,1))\}$. Then $e_1 = e_1^* = f_1 = (1,0)$, $e_2 = e_2^* = f_2 = (0,1)$, and

$$N = N^\circ = M = M^\circ = \langle (1,0),(0,1) \rangle,$$

Example

If we mutate in direction $k = 2$, we obtain:

$$\begin{aligned} e_1' &= (1,0) + 1[1]_+(0,1) = (1,1) \\ e_2' &= (0,-1) \\ f_1' &= (1,0) \\ f_2' &= -(0,1) + 1\left([1]_+(1,0) + [0]_+(0,1)\right) = (1,-1) \\ \epsilon_{12} &= -1 \\ \epsilon_{21} &= 1 \end{aligned}$$

And therefore $\mu_2(\mathbf{s}) = \{((1,1),(1,a,a,1)),((0,-1),(1,1))\}$.

Injectivity Assumption

To construct cluster scattering diagrams, we must assume that the map

$$\begin{aligned} p_1^* : N_{\mathrm{uf}} &\to M^\circ \\ n &\mapsto \{n, \cdot\} \end{aligned}$$

is injective.

This is known as the **injectivity assumption**.

Note: This is not true for all choices of fixed data, but is true in the principal coefficient case.

Generalized Cluster Varieties

Each choice of **s** defines algebraic tori

$$\mathcal{X}_{\mathbf{s}} = T_M(R) = T_M \times_{\Bbbk} \mathrm{Spec}(R) = \mathrm{Spec}\left(\Bbbk[N]\right) \times_{\Bbbk} \mathrm{Spec}(R),$$
$$\mathcal{A}_{\mathbf{s}} = T_{N^\circ}(R) = T_{N^\circ} \times_{\Bbbk} \mathrm{Spec}(R) = \mathrm{Spec}\left(\Bbbk[M^\circ]\right) \times_{\Bbbk} \mathrm{Spec}(R).$$

We define birational maps $\mu_k : \mathcal{X}_{\mathbf{s}} \to \mathcal{X}_{\mu_k(\mathbf{s})}$ and $\mu_k : \mathcal{A}_{\mathbf{s}} \to \mathcal{A}_{\mu_k(\mathbf{s})}$ via the pull-back of functions

$$\mu_k^* z^n = z^n \left(1 + a_{k,1} z^{e_k} + \cdots + a_{k,r_k-1} z^{(r_k-1)e_k} + z^{r_k e_k}\right)^{-[n,e_k]}$$
$$\mu_k^* z^m = z^m \left(1 + a_{k,1} z^{v_k} + \cdots + a_{k,r_k-1} z^{(r_k-1)v_k} + z^{r_k v_k}\right)^{-\langle d_k e_k, m\rangle}$$

for $n \in N$ and $m \in M^\circ$.

Generalized Cluster Varieties

The generalized $\mathcal{X}$ and $\mathcal{A}$ cluster varieties are the schemes

$$\mathcal{A} := \bigcup_{s \in \mathfrak{T}} \mathcal{A}_{\mathbf{s}}, \qquad \mathcal{X} := \bigcup_{\mathbf{s} \in \mathfrak{T}} \mathcal{X}_{\mathbf{s}}$$

where the collections $\{\mathcal{A}_{\mathbf{s}}\}_{\mathbf{s} \in \mathfrak{T}}$ and $\{\mathcal{X}_{\mathbf{s}}\}_{\mathbf{s} \in \mathfrak{T}}$ are glued according to the birational maps $\mu_k : \mathcal{A}_{\mathbf{s}} \to \mathcal{A}_{\mu_k(\mathbf{s})}$ and $\mu_k : \mathcal{X}_{\mathbf{s}} \to \mathcal{X}_{\mu_k(\mathbf{s})}$.

Basic Definitions

Given $\mathbf{s}$, set $N^+ := N_{\mathbf{s}}^+ := \left\{ \sum_{i \in I_{\mathrm{uf}}} a_i e_i \middle| a_i \geq 0, \sum a_i > 0 \right\}$.

Definition

A *wall* in $M_{\mathbb{R}}$ is a pair $(\mathfrak{d}, f_{\mathfrak{d}}) \in (N^+, \widehat{R[P]})$ such that for some primitive $n_0 \in N^+$,

1. $f_{\mathfrak{d}} \in \widehat{R[P]}$ has the form $1 + \sum_{j=1}^{\infty} c_j z^{jp_1^*(n_0)}$ with $c_j \in R$.
2. $\mathfrak{d} \subset n_0^{\perp} \subset M_{\mathbb{R}}$ is a $(\mathrm{rank}\ M - 1)$-dimensional convex rational polyhedral cone.

We refer to $\mathfrak{d} \subset M_{\mathbb{R}}$ as the *support* of the wall $(\mathfrak{d}, f_{\mathfrak{d}})$.

Definition

A *scattering diagram* $\mathfrak{D}$ for N^+ and $\mathbf{s}$ is a set of walls $\{(\mathfrak{d}, f_{\mathfrak{d}})\}$ such that for every degree $k > 0$, there are a finite number of walls $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}$ with $f_{\mathfrak{d}} \neq 1 \mod \mathfrak{m}^{k+1}$.

Constructing Cluster Scattering Diagrams

This construction begins with the *initial scattering diagram*

$$\mathfrak{D}_{\mathrm{in},\mathbf{s}} := \{(e_i^\perp, 1 + a_{i,1}z^{v_i} + a_{i,2}z^{2v_i} + \cdots + a_{i,r_i-1}z^{(r_i-1)v_i} + z^{r_i v_i}) : i \in I_{\mathrm{uf}}\}$$

where $v_i = \{e_i, \cdot\}$ for $i \in I_{\mathrm{uf}}$.

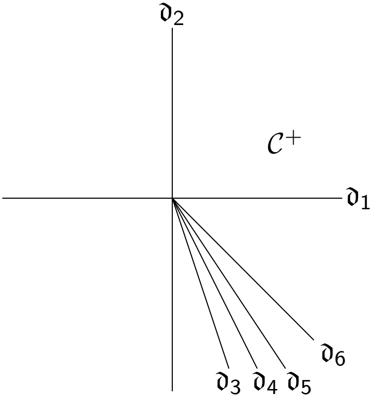
Theorem (Cheung-K.-Musiker, 2021)

Given a generalized torus seed $\mathbf{s}$*, there exists a consistent scattering diagram* $\mathfrak{D}_\mathbf{s}$ *such that* $\mathfrak{D}_{in,\mathbf{s}} \subset \mathfrak{D}_\mathbf{s}$ *and* $\mathfrak{D}_\mathbf{s} \setminus \mathfrak{D}_{in,\mathbf{s}}$ *consists only of walls* $\mathfrak{d} \subset n_0^\perp$ *with* $p_1^*(n_0) \notin \mathfrak{d}$*. The scattering diagram* $\mathfrak{D}_\mathbf{s}$ *is unique up to equivalence.*

Understanding this theorem statement requires a few more definitions.

Example

In our running example, we would construct the cluster scattering diagram:

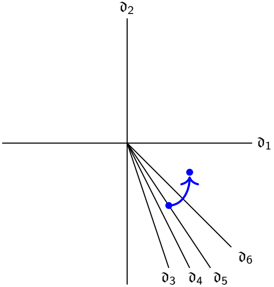


$$f_{\mathfrak{d}_1} = 1 + z^{(-1,0)}$$
$$f_{\mathfrak{d}_2} = 1 + az^{(0,1)} + az^{(0,2)} + z^{(0,3)}$$
$$f_{\mathfrak{d}_3} = 1 + z^{(-1,3)}$$
$$f_{\mathfrak{d}_4} = 1 + az^{(-1,2)} + az^{(-2,4)} + z^{(-3,6)}$$
$$f_{\mathfrak{d}_5} = 1 + z^{(-2,3)}$$
$$f_{\mathfrak{d}_6} = 1 + az^{(-1,1)} + az^{(-2,2)} + z^{(-3,3)}$$

Wall-crossing

Crossing a wall $(\mathfrak{d}, f_{\mathfrak{d}})$ acts on monomials as $z^m \mapsto z^m f_{\mathfrak{d}}^{\langle n_0, m\rangle}$, where n_0 is the primitive vector normal to $\mathfrak{d}$ that opposes the direction of travel.

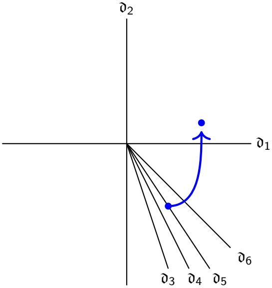
It acts on a polynomial by acting on each individual monomial.



$$
\begin{aligned}
z^{(2,-3)} &\mapsto z^{(2,-3)}\left(1 + az^{(-1,1)} + az^{(-2,2)} + z^{(-3,3)}\right)^{\langle (2,3),(-1,-1)\rangle} \\
&= z^{(2,-3)}\left(1 + az^{(-1,1)} + az^{(-2,2)} + z^{(-3,3)}\right)
\end{aligned}
$$

Path-ordered Products

Composing multiple wall-crossings along a path γ gives us the *path-ordered product* $\mathfrak{p}_\gamma$.

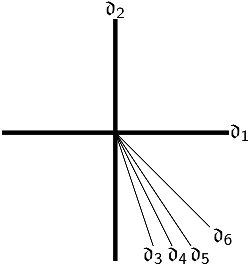


$$
\begin{aligned}
z^{(2,-3)} &\mapsto z^{(2,-3)}\left(1+az^{(-1,1)}+az^{(-2,2)}+z^{(-3,3)}\right)^{\langle(2,3),(-1,-1)\rangle}\\
&= z^{(2,-3)}\left(1+az^{(-1,1)}+az^{(-2,2)}+z^{(-3,3)}\right)\\
&\mapsto z^{(2,-3)}\left(1+z^{(-1,0)}\right)^3\cdot\\
&\qquad\left(1+\frac{az^{(-1,1)}}{1+z^{(-1,0)}}+\frac{az^{(-2,2)}}{(1+z^{(-1,0)})^2}+\frac{z^{(-3,3)}}{(1+z^{(-1,0)})^3}\right)\\
&= z^{(2,-3)}\left(\begin{array}{c}(1+z^{(-1,0)})^3+az^{(-1,1)}(1+z^{(-1,0)})^2\\ +az^{(-2,2)}(1+z^{(-1,0)})+z^{(-3,3)}\end{array}\right)
\end{aligned}
$$

Consistency and Equivalence

A scattering diagram $\mathfrak{D}$ is *consistent* if $\mathfrak{p}_\gamma = 1$ for any closed loop γ. We build $\mathfrak{D}_{\mathbf{s}}$ from $\mathfrak{D}_{\mathrm{in},\mathbf{s}}$ by adding any walls necessary to satisfy this condition.

In our running example, $\mathfrak{D}_{\mathrm{in}}$ consists of the vertical and horizontal walls.



Two scattering diagrams, $\mathfrak{D}$ and $\mathfrak{D}'$, are *equivalent* if $\mathfrak{p}_{\gamma,\mathfrak{D}} = \mathfrak{p}_{\gamma,\mathfrak{D}'}$ for all paths γ for which both path-ordered products are defined.

Mutation Invariance

Because mutation equivalent seeds $\mathbf{s}$ and $\mathbf{s}'$ generate the same generalized cluster algebra, we should expect $\mathfrak{D}_{\mathbf{s}}$ and $\mathfrak{D}_{\mathbf{s}'}$ to be equivalent.

Let

$$\mathcal{H}_{k,+} := \{m \in M_{\mathbb{R}} : \langle e_k, m\rangle \geq 0\},$$
$$\mathcal{H}_{k,-} := \{m \in M_{\mathbb{R}} : \langle e_k, m\rangle \leq 0\}.$$

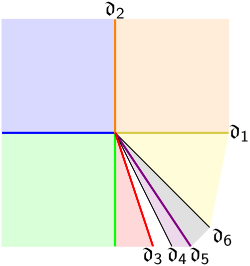
The birational maps $\mu_k : \mathcal{A}_{\mathbf{s}} \to \mathcal{A}_{\mu_k(\mathbf{s})}$ and $\mu_k : \mathcal{X}_{\mathbf{s}} \to \mathcal{X}_{\mu_k(\mathbf{s})}$ tropicalize to the piecewise linear map

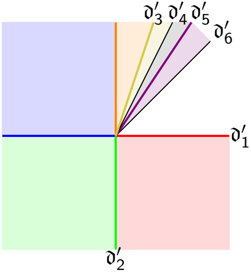
$$T_k(m) := \begin{cases} m + r_k v_k \langle d_k e_k, m\rangle & m \in \mathcal{H}_{k,+} \\ m & m \in \mathcal{H}_{k,-} \end{cases}$$

We compute $T_k(\mathfrak{D})$ by:

1. Replacing the wall $(e_k^\perp, 1 + a_{k,1}z^{v_k} + \cdots + a_{k,r_k-1}z^{(r_k-1)v_k} + z^{r_k v_k})$ with $(e_k^\perp, 1 + a_{k,1}z^{-v_k} + \cdots + a_{k,r_k-1}z^{-(r_k-1)v_k} + z^{-r_k v_k})$.
2. Applying T_k to the support and wall-crossing automorphism of each remaining wall.

Example: Applying T_1 to our running example, we get:





Note: T_k is only an involution up to equivalence of diagrams.

Mutation Invariance

Theorem (Cheung-K.-Musiker, 2021)

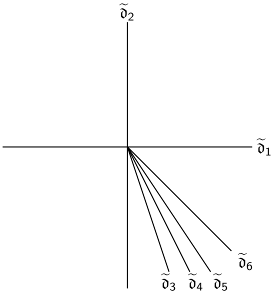
If the injectivity assumption holds, then $T_k(\mathfrak{D}_{\mathbf{s}})$ is a consistent scattering diagram for $N^+_{\mu_k(\mathbf{s})}$. Moreover, the diagrams $\mathfrak{D}_{\mu_k(\mathbf{s})}$ and $T_k(\mathfrak{D}_{\mathbf{s}})$ are equivalent.

The proof of this theorem is quite lengthy, but one key point is that it relies on the reciprocity condition $a_{k,i} = a_{k,r_k-i}$.

Principal Coefficients

All of these definitions and statements can also be made for the *principal coefficient case* (for details, see our preprint). In fact, our proofs proceed via this case because it necessarily satisfies the injectivity assumption.

Example: In our running example, $\mathfrak{D}_{\mathbf{s}_{\mathrm{prin}}}$ is



$$f_{\widetilde{\mathfrak{d}}_1} = 1 + z^{(-1,0,0,1)}$$
$$f_{\widetilde{\mathfrak{d}}_2} = 1 + az^{(0,1,1,0)} + az^{(0,2,2,0)} + z^{(0,3,3,0)}$$
$$f_{\widetilde{\mathfrak{d}}_3} = 1 + z^{(-1,3,3,1)}$$
$$f_{\widetilde{\mathfrak{d}}_4} = 1 + az^{(-1,2,2,1)} + az^{(-2,4,4,2)} + z^{(-3,6,6,3)}$$
$$f_{\widetilde{\mathfrak{d}}_5} = 1 + z^{(-2,3,3,2)}$$
$$f_{\widetilde{\mathfrak{d}}_6} = 1 + az^{(-1,1,1,1)} + az^{(-2,2,2,2)} + z^{(-3,3,3,3)}$$

Note: $\mathfrak{D}_{\mathbf{s}_{\mathrm{prin}}}$ is actually four-dimensional; this drawing is a projection.

Building $\mathcal{A}_{\mathrm{scat},\mathbf{s}}$ and showing $\mathcal{A}_{\mathrm{scat},\mathbf{s}} \cong \mathcal{A}_{\mathbf{s}}$

Given $\mathfrak{D}_{\mathbf{s}}$, we build a scheme $\mathcal{A}_{\mathrm{scat},\mathbf{s}}$ by associating tori to each chamber and gluing along the birational mutation maps. To show that the algebra of $\mathcal{A}_{\mathrm{scat},\mathbf{s}}$ is the associated generalized algebra, we show that $\mathcal{A}_{\mathrm{scat},\mathbf{s}} \cong \mathcal{A}_{\mathbf{s}}$.

A key step in doing so is checking the commutativity of the following diagram for mutation equivalent seeds $\mathbf{s}$ and $\mathbf{s}'$.

$$\begin{array}{ccc} T_{N^\circ,\sigma} & \xrightarrow{T_{v',\sigma}} & T_{N^\circ,\sigma'} \\ {\scriptstyle \mathfrak{p}_{\sigma,\tilde{\sigma}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{p}_{\sigma',\tilde{\sigma}'}} \\ T_{N^\circ,\tilde{\sigma}} & \xrightarrow[T_{v',\tilde{\sigma}}]{} & T_{N^\circ,\tilde{\sigma}'} \end{array}$$

where σ and $\widetilde{\sigma}$ are chambers in some $\mathfrak{D}_{\mathbf{s}}$ and σ' and $\widetilde{\sigma}'$ are the corresponding chambers in $\mathfrak{D}_{\mathbf{s}'}$.

This again requires the reciprocity condition.

Theorem (Cheung-K.-Musiker, 2021)

Let $\mathbf{s}$ *be a generalized torus seed,* v *be the root of* $\mathfrak{T}_{\mathbf{s}}$ *and* v' *be any other vertex. Let* $\psi^*_{v,v'} : M^\circ_{v'} \to M^\circ_{v'}$ *be the linear map* $\mu^T_{v,v'}\Big|_{\mathcal{C}^+_{v'\in\mathbf{s}}}$ *and* $\psi_{v,v'} : T_{N^\circ,v'} \to T_{N^\circ,v'}$ *be the map between the associated tori. Then the collection* $\{\psi_{v,v'}\}_{v'}$ *glue to give an isomorphism*

$$\mathcal{A}_{\mathbf{s}} := \bigcup_{v'} T_{N^\circ} \to \mathcal{A}_{scat,\mathbf{s}} := \bigcup_{v'} T_{N^\circ,v'}$$

and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}_{\mathbf{s}} & \longrightarrow & \mathcal{A}_{\mathrm{scat},\mathbf{s}} \\ \downarrow & & \downarrow \\ \mathcal{A}_{\mathbf{s}_{v'}} & \longrightarrow & \mathcal{A}_{\mathrm{scat},\mathbf{s}_{v'}} \end{array}$$

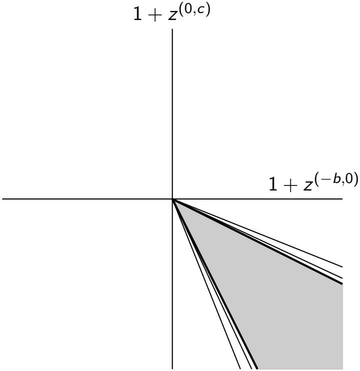
Upshot: We can identify the rings of regular functions on $\mathcal{A}_{\mathrm{scat},\mathbf{s}}$ and $\mathcal{A}_{\mathbf{s}}$.

Theta functions can be defined in terms of path-ordered products as

$$\vartheta_m = \mathfrak{p}_\gamma(z^m)$$

where γ is a path to the positive chamber.

These path-ordered products are easy to compute in areas where the walls aren't dense. In many scattering diagrams, though, there are dense regions:



When $bc \geq 5$, every wall with rational slope appears inside the shaded cone.

("The badlands")

Broken Lines

This motivates *broken lines*, which give another way to define ϑ_m.

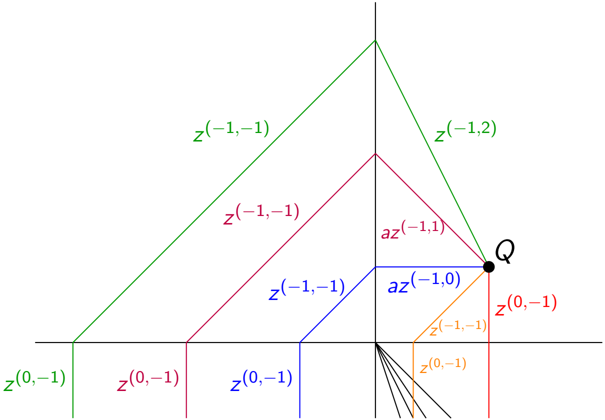
Roughly, a broken line is a collection of all piecewise linear paths which begin with slope $-m_0$, "scatter" off the walls in particular ways dictated by the wall-crossing automorphisms, and end at a particular point Q.

In terms of broken lines,

$$\vartheta_{Q,m_0} := \sum_{\gamma} \mathrm{Mono}(\gamma)$$

where the summation ranges over all broken lines γ with initial slope $-m_0$ and endpoint Q and $\mathrm{Mono}(\gamma)$ denotes the monomial attached to the final domain of linearity.

Example: Let $m_0 = (0, -1)$ and Q be below the diagonal.



$$\vartheta_{Q,(0,-1)} = z^{(0,-1)} + z^{(-1,-1)} + az^{(-1,0)} + az^{(-1,1)} + z^{(-1,2)}$$
$$= \frac{1 + x_1 + ax_2 + ax_2^2 + x_2^3}{x_1 x_2}$$

Some Results

Theorem (Cheung-K.-Musiker, 2021)

Let Γ be generalized fixed data that satisfies the injectivity assumption, $\mathbf{s}$ be a choice of generalized torus seed, Q be a point in $\mathcal{C}_{\mathbf{s}}^{+}$, and m be a point in $\sigma \cap M^{\circ}$ for some chamber $\sigma \in \Delta_{\mathbf{s}}^{+}$.

Then $\vartheta_{Q,m}$ expresses a cluster monomial of the associated generalized cluster algebra in $\mathbf{s}$ as a Laurent polynomial. Moreover, all cluster monomials can be expressed as $\vartheta_{Q,m}$ for some choice of Q and m.

Upshot: The generalized cluster monomials can be expressed in terms of theta functions.

Some Results

Theorem (Cheung-K.-Musiker, 2021)

Let Γ be a set of generalized fixed data and $\mathbf{s}$ be a choice of associated initial torus seed data, which defines the set of dual vectors $\{f_i = d_i^{-1} e_i^*\}$. If $\mathbf{s}'$ is mutation equivalent to $\mathbf{s}$, then the i-th coordinates of the g-vectors for the cluster variables in $\mathbf{s}'$ are either all non-negative or all non-positive when expressed in the basis $\{f_1, \ldots, f_n\}$.

Upshot: Sign-coherence of g-vectors.

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