Schubert polynomials and the inhomogeneous TASEP on a ring

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(joint work with Lauren Williams)

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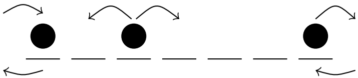
Asymmetric Exclusion Process

- The asymmetric exclusion process (ASEP) is an important model from statistical mechanics which describes a system of particles on a lattice hopping left and right.
- There are many variants of the ASEP and these were studied with various mathematical approaches, for example, Bethe Ansatz, quadratic algebras, combinatorics, orthogonal polynomials, random matrices, stochastic differential equations and hydrodynamic limits.
- The ASEP has many applications in a broad range including protein synthesis, traffic flow, formation shocks, surface growth, and sequence alignments.

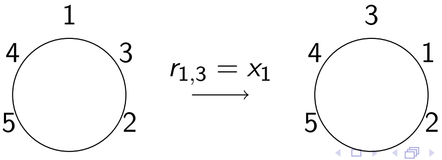


The inhomogenous TASEP definition

- Consider a lattice with n sites arranged in a ring. Let $St(n)$ denote the $n!$ labelings of the lattice by distinct numbers $1, 2, \ldots, n$, where each number i is called a *particle of weight* i.
- The *inhomogeneous TASEP on a ring of size* n is a Markov chain with state space $St(n)$ where at each time t a swap of two adjacent particles may occur: a particle of weight i on the left swaps its position with a particle of weight j on the right with transition rate $r_{i,j}$ given by:

$$r_{i,j} = \begin{cases} x_i & \text{if } i < j \\ 0 & \text{otherwise.} \end{cases}$$

- Eg.)



The inhomogeneous TASEP $n = 3$

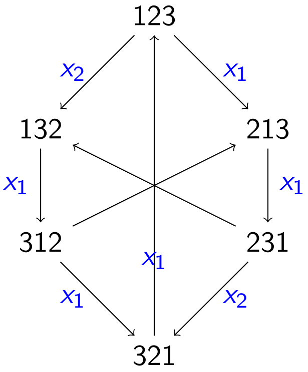


Figure: The transition diagram for the inhomogeneous TASEP for $n = 3$

Renormalized steady state probabilities

- The steady state probabilities for $n = 3$ inhomogeneous TASEP
  States 123, 231, 312: $\frac{x_1}{6x_1+3x_2}$

  States 132, 321, 213: $\frac{x_1+x_2}{6x_1+3x_2}$
- we multiply all steady state probabilities by the same constant, obtaining "renormalized" steady state probabilities ψ_w, so that

$$\psi_{123\ldots n} = \prod_{i<j} (x_i)^{j-i-1}.$$

- $n = 3$

$$\psi_{123} = \psi_{123} = \psi_{123} = x_1$$
$$\psi_{132} = \psi_{321} = \psi_{213} = x_1 + x_2$$

- Observation: ψ_w is a positive polynomial in x_i's.
- Schubert polynomials? $x_1 = \mathfrak{S}_{(2,1)}$, $x_1 + x_2 = \mathfrak{S}_{(1,3,2)}$

Schubert polynomials definition

Definition: Schubert polynomials

For the longest permutation $\sigma_0 \in S_n$

$$\mathfrak{S}_{\sigma_0}(x) = \prod_{1 \leq i \leq n} x_i^{n-i}$$

for generic $\sigma \in S_n$

$$\mathfrak{S}_{\sigma}(x) = \partial_{\sigma^{-1}\sigma_0} \mathfrak{S}_{\sigma_0}(x)$$

where $\partial_\sigma = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_l}$ ($s_{i_1} s_{i_2} \cdots s_{i_l}$ is a reduced decomposition of σ)

$$(\partial_i P)(x_1, \ldots, x_n) = \frac{P(\ldots, x_i, x_{i+1}, \ldots) - P(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}$$

Table $n = 4$

State w	Probability ψ_w
1234	$x_1^3 x_2$
1324	$x_1 \mathfrak{S}_{1432}$
1342	$x_1 x_2 \mathfrak{S}_{1423}$
1423	$x_1^2 x_2 \mathfrak{S}_{1243}$
1243	$x_1^2 \mathfrak{S}_{1342}$
1432	$\mathfrak{S}_{1423} \mathfrak{S}_{1342}$

Table $n = 5$,

State w	Probability ψ_w
12345	$\mathbf{x}^{(6,3,1)}$
12354	$\mathbf{x}^{(5,2,0)}\mathfrak{S}_{13452}$
12435	$\mathbf{x}^{(4,1,0)}\mathfrak{S}_{14532}$
12453	$\mathbf{x}^{(4,1,1)}\mathfrak{S}_{14523}$
12534	$\mathbf{x}^{(5,2,1)}\mathfrak{S}_{12453}$
12543	$\mathbf{x}^{(3,0,0)}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$
13245	$\mathbf{x}^{(3,1,1)}\mathfrak{S}_{15423}$
13254	$\mathbf{x}^{(2,0,0)}\mathfrak{S}_{15423}\mathfrak{S}_{13452}$
13425	$\mathbf{x}^{(3,2,1)}\mathfrak{S}_{15243}$
13452	$\mathbf{x}^{(3,3,1)}\mathfrak{S}_{15234}$
13524	$\mathbf{x}^{(2,1,0)}(\mathfrak{S}_{164325} + \mathfrak{S}_{25431})$
13542	$\mathbf{x}^{(2,2,0)}\mathfrak{S}_{15234}\mathfrak{S}_{13452}$
14235	$\mathbf{x}^{(4,2,0)}\mathfrak{S}_{13542}$
14253	$\mathbf{x}^{(4,2,1)}\mathfrak{S}_{12543}$
14325	$\mathbf{x}^{(1,0,0)}(\mathfrak{S}_{1753246} + \mathfrak{S}_{265314} + \mathfrak{S}_{2743156} + \mathfrak{S}_{356214} + \mathfrak{S}_{364215} + \mathfrak{S}_{365124})$
14352	$\mathbf{x}^{(1,1,0)}\mathfrak{S}_{15234}\mathfrak{S}_{14532}$
14523	$\mathbf{x}^{(4,3,1)}\mathfrak{S}_{12534}$
14532	$\mathbf{x}^{(1,1,1)}\mathfrak{S}_{15234}\mathfrak{S}_{14523}$
15234	$\mathbf{x}^{(5,3,1)}\mathfrak{S}_{12354}$
15243	$\mathbf{x}^{(3,1,0)}(\mathfrak{S}_{146325} + \mathfrak{S}_{24531})$
15324	$\mathbf{x}^{(2,1,1)}(\mathfrak{S}_{15432} + \mathfrak{S}_{164235})$
15342	$\mathbf{x}^{(2,2,1)}\mathfrak{S}_{15234}\mathfrak{S}_{12453}$
15423	$\mathbf{x}^{(3,2,0)}\mathfrak{S}_{12534}\mathfrak{S}_{13452}$
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Positivity Conjectures

Conjecture1

The steady state probability ψ_w is a positive polynomial in x_i's.

Conjecture2

The steady state probability ψ_w is a positive sum Schubert polynomials.

- Lam and Williams in 2010 studied this model and made the above conjectures.
- Conjecture 1 has been proved by Arita and Mallick in 2012 by giving a monomial expansion formula in terms of multiline queues as conjectured by Ayyer and Linusson. (x, y version is still open)
- Later, Cantini in 2016 generalized the model by putting y-parameters.

Main results

Definition(K, Williams)

We say that $w \in S_n$ is a *evil-avoiding*, if: $w_1 = 1$; w avoids the patterns 2413, 3214, 4132, and 4213. We say $w \in St(n,k)$ if w is evil-avoiding and w^{-1} has exactly k *descents*.

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Theorem(K, Williams)

For $w \in St(n,k)$, the steady-state probability ψ_w is given as a trivial factor times product of k Schubert polynomials.

Eg) $w = (1,2,5,4,3)$, $w^{-1} = (1,2,5,4,3)$.
$w \in St(5,2)$.

$$\psi_w = x_1^3 \mathfrak{S}_{(1,4,5,2,3)} \mathfrak{S}_{(1,3,4,5,2)}$$

Main results

$\rightarrow$ The number of evil-avoiding permutations in S_n is $\frac{(2+\sqrt{2})^{n-1}+(2-\sqrt{2})^{n-1}}{2}$. Previously the formula (explaining the appearance of (double) Schubert polynomials) for n out of $n!$ states were known by Cantini.

- Cantini gave an explicit formula for ψ_w for the permutation w of the form $w(n,h) := (1, h+1, h+2, \ldots, n, h, h-1, \ldots, 2)$
- We gave an explicit formula for ψ_w for evil-avoiding permutations.
- We introduced z-Schubert polynomials $\mathfrak{S}^n_\lambda(z;x)$ for our main tool.

Thanks for your attention!

https://arxiv.org/abs/2106.13378

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