Fermionic Diagonal Coinvariants and Exterior Lefschetz Elements

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Motivation

- Delta (prime) operator $\Delta'_F : \Lambda_n \to \Lambda_n$ acts on the modified Macdonald basis $\{\tilde{H}_\mu\}$.
- (Delta Conjecture) Haglund, Remmel and Wilson conjectured in 2015 a generalization of the Shuffle conjecture
$$\Delta'_{e_{k-1}} e_n = \mathrm{Rise}_{n,k}(X;q,t) = \mathrm{Val}_{n,k}(X;q,t)$$
- The first equality is proven recently by (D'Adderio, Mellit) and (Blasiak, Haiman, Morse, Pun, Seelinger) independently.
- In 2019, Zabrocki proposed a module for the delta conjecture introducing the Fermionic variables
- The Boson-Fermionic generalization of the diagonal coinvariants:

  $S_{k,j,n} := \mathbb{C}[X_{k\times n}] \otimes \wedge\{\Theta_{j\times n}\}$

  $I_{k,j,n} := \langle f \in S_{k,j,n} \mid w.f = f$ for all $w \in S_n$ and constant$(f) = 0\rangle$ under multi-diagonal action of S_n

  $R_{k,j,n} := S_{k,j,n}/I_{k,j,n}$
- Some specializations:
  - $R_{1,0,n}$: classical coinvariant algebera
  - $R_{2,0,n}$: the diagonal coinvariant DR_n.
  - $R_{0,1,n}$: simple quotient of dimension 2^{n-1}.
  - $R_{2,1,n}$: Zabrocki's module for the Delta conjecture.

Fermionic diagonal coinvariants

- $FDR_n = R_{0,2,n}$, "fermionic" version of the diagonal coinvariants
- More generally let W be a complex reflection group of rank n acting irreducibly on its reflection representation $V = \mathbb{C}^n$.
- We consider the diagonal action of W on $\wedge(V \oplus V^*) \cong \oplus_{i,j}(\wedge^i V \otimes \wedge^j V^*)$.
- Similarly, let $\wedge(V \oplus V^*)^W_+$ be the subspace of the W-invariants with vanishing constant term.
- The W-fermionic diagonal coinvariants
$$\begin{aligned} FDR_W &:= \wedge(V \oplus V^*)/\langle \wedge(V \oplus V^*)^W_+ \rangle \\ &= \oplus_{i,j=0}^n (FDR_W)_{i,j} \\ &= \wedge\{\Theta_n, \Xi_n\}/\langle \wedge\{\Theta_n, \Xi_n\}^W_+ \rangle \end{aligned}$$

Dimensions

- Catalan numbers and Narayana numbers
$$\mathrm{Cat}(n) := \frac{1}{n+1}\binom{2n}{n} \qquad \mathrm{Nar}(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$
- **Theorem** (Kim, Rhoades)
  - $\dim(FDR_W) = \binom{2n+1}{n}$
  - $(FDR_W)_{i,j} = 0$ for $i+j > n$
  - For $i+j \leq n$, $\dim(FDR_W)_{i,j} = \binom{n}{i}\binom{n}{j} - \binom{n}{i-1}\binom{n}{j-1}$.
  - $\dim(FDR_W)_{k,n-k} = \mathrm{Nar}(n,k)$
  - $\sum_{k=0}^n \dim(FDR_W)_{k,n-k} = \mathrm{Cat}(n)$

Lefschetz Property

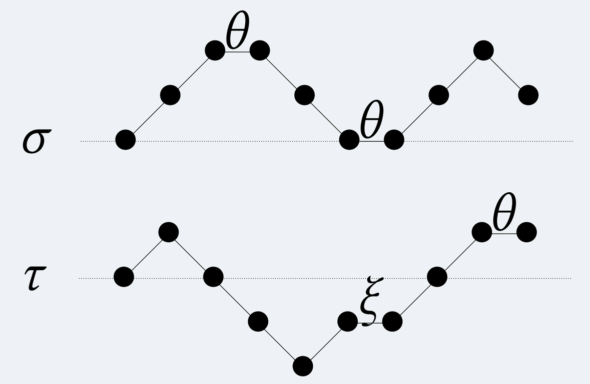
- The Casimir element $\delta_n := \theta_1\xi_1 + \cdots + \theta_n\xi_n \in \wedge\{\Theta_n, \Xi_n\}_{1,1}$
- (Hard Lefschetz Property) Suppose $i + j \leq n$ and let $r = n - i - j$. The following linear map is bijective
$$\phi : \wedge\{\Theta_n, \Xi_n\}_{i,j} \xrightarrow{\delta_n^r \cdot ()} \wedge\{\Theta_n, \Xi_n\}_{n-j,n-i}$$
- The key idea is to choose the bases wisely: let $A, B \subset [n]$ and
$$\begin{aligned} A - B &= \{a_1 < \cdots < a_r\} \\ B - A &= \{b_1 < \cdots < b_s\} \\ A \cap B &= \{c_1 < \cdots < c_t\} \\ v(A,B) &= (\theta_{c_1}\xi_{c_1} \cdots \theta_{c_t}\xi_{c_t})(\theta_{a_1} \cdots \theta_{a_r})(\xi_{b_1} \cdots \xi_{b_s}) \end{aligned}$$
  and use a similar result of (Hara, Watanabe) on the HLP of the cohomology ring of the n-fold product $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$: combinatorially, a result on the invertibility of the incidence matrix for the Boolean poset $B(n)$.

The Casimir element

- In the general case of W, define $\delta_W := \delta_n = p_1[\Theta\Xi_n]$
- The Casimir element δ_W generates the W-invariant subspace $\wedge(V \oplus V^*)^W_+$
- **Theorem** (Kim, Rhoades) In the Grothendieck ring of W,
$$[(FDR_W)_{i,j}] = [\wedge^i V] \cdot [\wedge^j V^*] - [\wedge^{i-1} V] \cdot [\wedge^{j-1} V^*]$$
- The key idea is that the multiplication by the Casimir element is a W-equivariant injection in the degree $i + j \leq n$ and a W-equivariant surjection in the higher degree.

Monomial basis : Motzkin paths

- A variant of Motzkin paths with decorated horizontal steps: Lattice paths from $(0,0)$ consisting of steps $(1,1), (1,0)^\theta, (1,0)^\xi, (1,-1)$.
$$\Pi(n) = \{\sigma = (s_1, \ldots, s_n) \mid s_i \in \{(1,1), (1,-1), (1,0)^\theta, (1,0)^\xi\}\}$$
- The depth $d(\sigma)$ = minimum y-coordinate reached
- The weight $\mathrm{wt}(\sigma) = \prod_i \mathrm{wt}(s_i)$ where
$$\mathrm{wt}(s_i) = 1, \text{ if } s_i = (1,1), \quad \mathrm{wt}(s_i) = \theta_i, \text{ if } s_i = (1,0)^\theta$$
$$\mathrm{wt}(s_i) = \xi_i, \text{ if } s_i = (1,0)^\xi, \quad \mathrm{wt}(s_i) = \theta_i\xi_i, \text{ if } s_i = (1,-1)$$
- The degree of a path
  deg(σ) = deg(wt((σ))) = n – terminal y-coordinate
- Lexicographical order on the steps and so on paths:
$$(1,1) <^{lex} (1,0)^\theta <^{lex} (1,0)^\xi <^{lex} (1,-1)$$
  Total order on paths / monomials imposed using the above definitions.
- **Theorem**(Kim, Rhoades) Let $\Pi(n)_{\geq 0} = \{\sigma \in \Pi(n) \mid d(\sigma) = 0\}$. The set of monomials $\{\mathrm{wt}(\sigma) \mid \sigma \in \Pi(n)_{\geq 0}\}$ is the standard monomial basis of FDR_W.
- Non-commutative Groebner basis.
- Example:



$$\mathrm{wt}(\sigma) = \theta_3 \cdot \theta_4\xi_4 \cdot_5 \xi_5 \cdot \theta_6 \cdot \theta_9\xi_9 \quad d(\sigma) = 0$$
$$\mathrm{wt}(\tau) = \theta_2\xi_2 \cdot \theta_3\xi_3 \cdot \theta_4\xi_4 \cdot \xi_6 \cdot \theta_9 \quad d(\tau) = -2$$

S_n-case

- Originanally for its permutation representation $U = V \oplus U^{S_n}$
- We have $\wedge(U \otimes U^*)/\langle \wedge(U \otimes U^*)^{S_n}_+ \rangle \cong \wedge(V \otimes V^*)/\langle \wedge(V \otimes V^*)^{S_n}_+ \rangle$
- In terms of anti-commuting polynomials, $I_{0,2,n} = \langle p_1[\Theta_n], p_1[\Xi_n], p_1[\Theta\Xi_n] \rangle$, we take successive quotients.