Geometric vertex decomposition and liaison

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(based on joint work with Jenna Rajchgot)

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Two related viewpoints

Lots of us like a lot of the same ideals and varieties:

- ideals of $k \times k$ minors of a generic $m \times n$ matrix $\leftrightarrow$ open patch of a Grassmannain Schubert variety
- one-sided mixed ladder determinantal ideals $\leftrightarrow$ Schubert determinantal ideals for vexillary (i.e. 2143-avoiding) permutations
- two-sided mixed ladder determinantal ideals $\leftrightarrow$ certain Kazhdan-Lusztig ideals
- varieties of complexes $\leftrightarrow$ other Kazhdan-Lusztig ideals

Today's goal: to describe how the techniques employed on the two sides of these correspondences are related.

Geometric vertex decomposition (Knutson–Miller–Yong)

Let $I = \langle x_{11}x_{22} - x_{21}x_{12}, x_{11}x_{23} - x_{21}x_{13}, x_{12}x_{23} - x_{13}x_{22}\rangle$. Let $<$ be Lex with $y = x_{23}$ largest.

Example

$$\begin{aligned} \mathrm{in}_y I &= \langle x_{11}x_{23}, x_{12}x_{23}, x_{11}x_{22} - x_{21}x_{12}\rangle \\ &= \langle x_{11}x_{22} - x_{21}x_{12}, x_{23}\rangle \cap \langle x_{11}, x_{12}\rangle = (N_{y,I} + \langle y\rangle) \cap C_{y,I}. \end{aligned}$$

Example

Let $J = \mathrm{in}_< I = \langle x_{11}x_{22}, x_{11}x_{23}, x_{12}x_{23}\rangle$.
$\mathrm{in}_y J = J = \langle x_{11}x_{22}, x_{23}\rangle \cap \langle x_{11}, x_{22}\rangle = (N_{y,J} + \langle y\rangle) \cap C_{y,J}$.
In the monomial case, a geometric vertex decomposition recovers a vertex decomposition of a simplicial complex.

Geometrically vertex decomposable ideals

An ideal $I \subseteq \kappa[x_1, \dots, x_n]$ is geometrically vertex decomposable if it is unmixed and if

1. $I = \langle 1 \rangle$ or I is generated by indeterminates, or
2. for some $y = x_i$, we have $\text{in}_y I = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ is a geometric vertex decomposition with $N_{y,I}$ and $C_{y,I}$ geometrically vertex decomposable.

Examples: vertex decomposable Stanley-Reisner ideals, determinantal ideals, ladder determinantal ideals, Schubert determinantal ideals, Kazhdan-Lusztig ideals, defining ideals of lower bound cluster algebras, ...

Theorem (K.-Rajchgot): If I is geometrically vertex decomposable, then $I = \sqrt{I}$ and $\kappa[x_1, \dots, x_n]/I$ is Cohen–Macaulay.

Elementary G-biliaison and Gröbner bases

Theorem (Gorla-Migliore-Nagel)

If $I' \subseteq in_< I$ and we have the two isomorphisms $I/N \cong [C/N](-\ell)$ and $I'/in_<(N) \cong [in_<(C)/in_<(N)](-\ell)$, then $I' = in_< I$.

Example

If $I = \langle x_{11}x_{22} - x_{21}x_{12}, x_{11}x_{23} - x_{21}x_{13}, x_{12}x_{23} - x_{13}x_{22} \rangle$, $N = \langle x_{11}x_{22} - x_{21}x_{12} \rangle$, and $C = \langle x_{11}, x_{12} \rangle$, then we have

$$I/N \xrightarrow{\frac{x_{11}x_{23} - x_{21}x_{13}}{x_{11}}} C/N \text{ and}$$

$$\langle x_{11}, x_{12} \rangle / \langle x_{11}x_{22} \rangle \xrightarrow{x_{23}} \langle x_{11}x_{23}, x_{12}x_{23}, x_{11}x_{22} \rangle / \langle x_{11}x_{22} \rangle$$

Theorem (K.-Rajchgot)

A homogeneous, saturated, unmixed ideal is geometrically vertex decomposable if and only if it is linked to a complete intersection via a series of elementary G-biliaisons of degree 1 with isomorphisms of particular forms.

$\text{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle) \leftrightarrow C_{y,I}/N_{y,I} \cong I/N_{y,I}$ with the isomorphism given by $\cdot f/g$ satisfying $\text{in}_y(f)/g = y$.

Payoffs

- The following ideals are easily seen to be glicci: Schubert determinantal ideals, Kazhdan–Lusztig ideals (including ideals defining varieties of complexes), ideals of graded lower bound cluster algebras
- A streamlined way to implement Migliore-Nagel-Gorla's strategy to establish Gröbner bases

Corollary (K.-Rajchgot)

Suppose $I = \langle yq_1 + r_1, \ldots, yq_k + r_k, h_1, \ldots, h_\ell \rangle$ *and* $\mathcal{G}_C = \{q_1, \ldots, q_k, h_1, \ldots, h_\ell\}$ *and* $\mathcal{G}_N = \{h_1, \ldots, h_\ell\}$ *are Gröbner bases. For suitable term orders, if the ideal of 2-minors of* $\begin{pmatrix} q_1 & \cdots & q_k \\ r_1 & \cdots & r_k \end{pmatrix}$ *is contained in* $N_{y,I}$ *and* $ht(N) < ht(I), ht(C)$, *then the given generators of* I *are Gröbner.*

The end

Thank you!

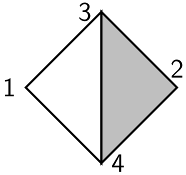
Squarefree monomial ideals

Recall the Stanley-Reisner correspondence between a squarefree monomial ideal $I_\Delta \subseteq \kappa[x_1, \ldots, x_n]$ and a simplicial complex Δ on vertices $x_1, \ldots, x_n$:

$$x_{i_1} \cdots x_{i_r} \in I_\Delta \iff \{i_1, \ldots, i_r\} \notin \Delta.$$

Example

$I_\Delta = \langle x_1x_2, x_1x_3x_4 \rangle = \langle x_1 \rangle \cap \langle x_2, x_3 \rangle \cap \langle x_2, x_4 \rangle \subseteq \kappa[x_1, \ldots, x_4]$

$\Delta =$ 

Because $\dim(\{2,3,4\}) = 2$ and $\dim(\{1,3\}) = 1$, Δ has two maximal faces that are not of the same dimension, and so Δ is not pure.

From now on, all simplicial complexes will be pure and all ideals unmixed.

Vertex decomposition

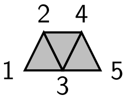
Definition

Given a simplicial complex Δ and a vertex v of Δ, define

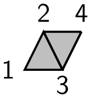
- $\mathrm{del}_\Delta(v) := \{F \in \Delta \mid F \cap \{v\} = \varnothing\}$. deletion of v
- $\mathrm{lk}_\Delta(v) := \{F \in \Delta \mid F \cup \{v\} \in \Delta, F \cap \{v\} = \varnothing\}$. link of v

Example

$I_\Delta = \langle x_1x_4, x_1x_5, x_2x_5 \rangle \subseteq \kappa[x_1, \ldots, x_5]$. $\Delta =$



$I_{\mathrm{del}_\Delta(5)} = \langle x_1x_4, x_5 \rangle \subseteq \kappa[x_1, \ldots, x_5]$. $\mathrm{del}_\Delta(5) =$



$I_{\mathrm{lk}_\Delta(5)} = \langle x_1, x_2, x_5 \rangle \subseteq \kappa[x_1, \ldots, x_5]$. $\mathrm{lk}_\Delta(5) =$

Vertex decomposition

Definition

A simplicial complex Δ is vertex decomposable if it is pure and if

1. Δ is a simplex or $\Delta = \varnothing$; or
2. $\exists$ vertex $v \in \Delta$ s.t. $\mathrm{lk}_\Delta(v)$ and $\mathrm{del}_\Delta(v)$ are vertex decomposable and $\dim(\Delta) = \dim(\mathrm{del}_\Delta(v)) = \dim(\mathrm{lk}_\Delta(v)) + 1$.

An important usage: If Δ is vertex decomposable, then Δ is Cohen–Macaulay, in which case I_Δ defines a Cohen–Macaulay variety.

Gröbner bases

Definition

If I is an ideal of the polynomial ring $S = \kappa[x_1, \ldots, x_n]$ equipped with a term order $<$, then a Gröbner basis for I is a generating set $\mathcal{G} = \{g_1, \ldots, g_k\}$ so that $\text{in}_< I = \langle \text{in}_<(g_1), \ldots, \text{in}_<(g_k) \rangle$.

Example

If $I = \langle xy, xz - w^2 \rangle$ and $<$ is the Lex with respect to $x > y > z > w$, then $\{xy, xz - w^2\}$ is *not* a Gröbner basis because $yw^2 = z(xy) - y(xz - w^2) \in I$, but $yw^2 = \text{in}_<(yw^2) \notin \langle xy, xz \rangle$. However, $\{xy, xz - w^2, yw^2\}$ is a Gröbner basis.

Some uses: Ideal membership questions, computations in Macaulay2, $\text{in}_< I$ radical or Cohen–Macaulay implies I radical or Cohen–Macaulay

Geometric vertex decomposition

Knutson-Miller-Yong '09: Let $S = \kappa[x_1, \ldots, x_n]$, $y = x_i$, and $<$ a lexicographic monomial with y largest.

- Let $\mathcal{G} = \{yq_1 + r_1, \ldots, yq_k + r_k, h_1, \ldots, h_\ell\}$ be a Gröbner basis for the ideal I with respect to $<$.
- Define $C_{y,I} := \langle q_1, \ldots, q_k, h_1, \ldots, h_\ell \rangle$ and $N_{y,I} := \langle h_1, \ldots, h_\ell \rangle$. $C_{y,I}$ will play the role of the link and $N_{y,I}$ of the deletion.
- Let $\mathrm{in}_y I := \langle yq_1, \ldots, yq_k, h_1, \ldots, h_\ell \rangle$.
- Then $\mathrm{in}_y I = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$, and this decomposition is called a geometric vertex decomposition of I with respect to y.
- Observation: $\mathrm{ht}(I) = \mathrm{ht}\ \mathrm{in}_y I = \mathrm{ht}(C_{y,I}) = \mathrm{ht}(N_{y,I}) + 1$.

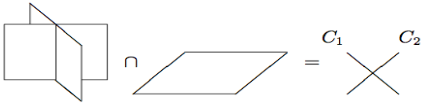
Motivation:

- geometry of Lex order "looks like" vertex decomposition
- useful for studying classes of generalized determinantal ideals

Liaison

Let C_1 and C_2 be equidimensional subschemes of $\mathbb{P}^n$. Liaison theory asks: if $X = C_1 \cup C_2$ is "nice", do good properties of C_1 transfer to C_2?

Example 2.4. If X is the complete intersection in $\mathbb{P}^3$ of a surface consisting of the union of two planes with a surface consisting of one plane then X links a line C_1 to a different line C_2.



From Migliore-Nagel's "Liaison and related topics."
For us today, "nice (enough)" will mean that C_1 and C_2 share no common component and that X is Gorenstein. One way to show that schemes are in the same Gorenstein biliaison class is via elementary G-biliaison.

Elementary G-biliaison

Definition

An ideal is glicci if it is in the Gorenstein liaison class of a complete intersection.

Fact: Glicci $\implies$ Cohen–Macaulay Question: Cohen–Macaulay $\implies$ Glicci?

Theorem (Hartshorne)

Suppose I, C, and N are homogeneous, saturated, unmixed ideals with $N \subseteq I \cap C$, $ht(N) + 1 = ht(I) = ht(C)$, and N Cohen–Macaulay and G_0. If $I/N \cong [C/N](-\ell)$ as graded R/N-modules, then I and C are evenly G-linked.

Whose initial ideals has elementary G-biliaison been used to study? Two-sided mixed ladder determinantal ideals (Gorla); Pfaffians of mixed size in a symmetric ladder, symmetric mixed ladder determiantal ideals (Gorla-Migliore-Nagel); Double determinantal varieties (Fieldsteel-K.); Rees algebras (Celikbas-Dufresne-Fouli-Gorla-Lin-Polini-Swanson),...