Some natural extensions of the parking space

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Parking functions

An integer sequence $(x_1, \ldots, x_n)$ is a *parking function* if its weakly increasing rearrangement $(z_1, \ldots, z_n)$ satisfies $0 \leq z_i \leq i - 1$ for $i = 1, \ldots, n$.

We denote by PF_n the set of all parking functions of length n.

$$\begin{aligned}
\mathrm{PF}_2 &= \{00, 01, 10\} \\
\mathrm{PF}_3 &= \{000, 001, 010, 100, 002, 020, 200, 011, \\
&\quad 101, 110, 012, 021, 102, 120, 201, 210\}
\end{aligned}$$

Motivation: cars, parking lots, preferences

Enumeration

Theorem (Pollak)

The map $\mathrm{PF}_n \to \mathbb{Z}_{n+1}^{n-1}$, given by

$$(x_1, \ldots, x_n) \mapsto (x_2 - x_1, \ldots, x_n - x_{n-1}),$$

is a bijection.

Pollak's theorem implies that $|PF_n| = (n+1)^{n-1}$.

The theorem says that for a sequence $(\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Z}_{n+1}^{n-1}$, exactly one of the sequences $(y, y+\alpha_1, y+\alpha_1+\alpha_2, \ldots, y+\alpha_1+\cdots+\alpha_{n-1})$, $y \in \mathbb{Z}_{n+1}$, is in PF_n.

Action

Rearranging the entries in one parking function results in another.

That means that we have an action (or representation) ρ_n of the symmetric group S_n on PF_n.

Recall that a (matrix) representation is completely determined by its *character* (trace). The character of a group element acting on a set is the number of fixed points.

Character

Take a permutation π with cycle type $(\lambda_1, \ldots, \lambda_\ell)$. Without loss of generality, we can assume

$$\pi = (1, \ldots, \lambda_1)(\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2) \cdots (\lambda_1 + \cdots + \lambda_{\ell-1} + 1, \ldots, n).$$

We want to count all $x \in \mathrm{PF}_n$ satisfying $\pi \cdot x = x$. Then $x_1 = \ldots = x_{\lambda_1}$, $x_{\lambda_1+1} = \ldots = x_{\lambda_1+\lambda_2}$ etc.

So we can arbitrarily choose $x_{\lambda_1+1} - x_{\lambda_1}, x_{\lambda_1+\lambda_2+1} - x_{\lambda_1+\lambda_2}, \ldots$, and that means that

$$\chi_{\rho_n}(\pi) = (n+1)^{\ell-1}.$$

Frobenius characteristic

For a representation ρ of S_n, we define the *Frobenius characteristic*

$$\mathrm{Frob}(\rho) = \frac{1}{n!} \sum_{\pi \in S_n} \chi_\rho(\pi) p_{\lambda(\pi)},$$

where p_λ is the *power sum symmetric function*.

Under Frob, the irreducible representation of S_n corresponding to the partition $\mu \vdash n$ gets mapped to the *Schur function* s_μ.

The Frobenius characteristic of a representation is therefore s-positive.

Action on permutations of a multiset

For the natural action of S_n on the set of permutations of the multiset $\{1^{a_1}, 2^{a_2}, \ldots, m^{a_m}\}$, the Frobenius characteristic is the *complete homogeneous symmetric* function $h_{a_1 \ldots a_m}$.

If the Frobenius characteristic of a matrix representation is h-positive, it is isomorphic to the permutation action on the disjoint union of sets of all permutations of some multisets.

Postnikov–Shapiro representation

Given a graph G on n vertices, we attach to it the polynomial $p(G) = \prod_{ij \in E(G)} (x_i - x_j) \in \mathbb{C}[x_1, \ldots, x_n]$.

Define V_n to be the $\mathbb{C}$-linear span of $p(G)$ over all G for which the complement $\overline{G}$ is a connected graph (i.e., G is a *slim graph*).

The natural action of S_n on $\mathbb{C}[x_1, \ldots, x_n]$ that permutes variables gives s representation σ_n on V_n because relabeling vertices preserves connectedness.

Restriction to S_{n-1}

Theorem (Berget–Rhoades)

The restriction of σ_n to S_{n-1} is isomorphic to ρ_{n-1}.

$\widehat{\mathrm{PF}}_{n,c}$

For $n \in \mathbb{N}$ and $1 \leq c \leq n$, define the set

$$\widehat{\mathrm{PF}}_{n,c} = \{(x_1, \ldots, x_n) \in \mathbb{Z}_n^n \colon (x_1, \ldots, x_{n-1}) \in \mathrm{PF}_{n-1}, \sum_{1 \leq i \leq n} x_i = c \pmod{n}\}.$$

$$\widehat{\mathrm{PF}}_{3,1} = \{001, 010, 100\}$$
$$\widehat{\mathrm{PF}}_{3,2} = \{002, 011, 101\}$$
$$\widehat{\mathrm{PF}}_{3,3} = \{000, 012, 102\}$$

The projection $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$ is a bijection $\widehat{\mathrm{PF}}_{n,c} \to \mathrm{PF}_{n-1}$. In particular, we have $|\widehat{\mathrm{PF}}_{n,c}| = n^{n-2}$.

Action on $\widehat{\mathrm{PF}}_{n,c}$

We can construct an action $\tau_{n,c}$ of S_n on $\widehat{\mathrm{PF}}_{n,c}$.

Take $\pi \in S_n$ and $(x_1, \ldots, x_n) \in \widehat{\mathrm{PF}}_{n,c}$. Note that $(x_{\pi_1}, \ldots, x_{\pi_{n-1}})$ is not necessarily in PF_{n-1}, and therefore $(x_{\pi_1}, \ldots, x_{\pi_n})$ is not necessarily in $\widehat{\mathrm{PF}}_{n,c}$.

However, by Pollak's theorem, exactly one of the sequences $(y + x_{\pi_1}, \ldots, y + x_{\pi_{n-1}})$ is in PF_{n-1}, and therefore $(y + x_{\pi_1}, \ldots, y + x_{\pi_n}) \in \widehat{\mathrm{PF}}_{n,c}$. This element is the action of π on $(x_1, \ldots, x_n)$.

Example

For example, consider the action of $\pi = 1432 \in S_4$ on $0003 \in \widehat{\mathrm{PF}}_{4,3}$.

Naïvely permuting elements of the sequence 0003 according to π leads to 0300.

Note that $030 \notin \mathrm{PF}_3$, but adding 1 to each coordinate gives $101 \in \mathrm{PF}_3$. Thus $1432 \cdot 0003 = 1011$.

Main result

Theorem

The map $\tau_{n,c}$ is an action of S_n on $\widehat{\mathrm{PF}}_{n,c}$ whose restriction to S_{n-1} is isomorphic to ρ_{n-1}. Furthermore, the character $\chi_{n,c} = \chi_{\tau_{n,c}}$ can be computed as follows. Choose a permutation $\pi \in S_n$ with cycle type $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, and write $d = \mathrm{GCD}(\lambda_1, \ldots, \lambda_\ell)$. Then

$$\chi_{n,c}(\pi) = \begin{cases} \frac{d^2 n^{\ell-2}}{2} & d \text{ even}, \frac{n}{d} \text{ odd, and } d|2c \\ d^2 n^{\ell-2} & d \text{ even}, \frac{n}{d} \text{ even, and } d|c \\ d^2 n^{\ell-2} & d \text{ odd and } d|c \\ 0 & \text{otherwise.} \end{cases}$$

Main result

Corollary

Choose a permutation $\pi \in S_n$ with cycle type $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, and write $d = \mathrm{GCD}(\lambda_1, \ldots, \lambda_\ell)$. Then

$$\chi_{n,1}(\pi) = \begin{cases} n^{\ell-2} & d = 1 \\ 2n^{\ell-2} & d = 2,\ n = 2 \pmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, the number of orbits of the action $\tau_{n,1}$ is given by

$$o_{n,1} = \frac{1}{n^2} \sum_{d|n} (-1)^{n+d} \mu(n/d) \binom{2d-1}{d}.$$

We can classify when $\tau_{n,c}$ and $\tau_{n,c'}$ are isomorphic.

Connection with Berget–Rhoades

Numerical data suggests that $\tau_{n,1}$ is isomorphic to the representation σ_n.

Also, it seems that $\mathrm{Frob}(\tau_{n,1})$ expands positively in the basis of homogeneous symmetric functions, i.e., it is *h-positive*.

Joint work with Sulzgruber

In joint work with Sulzgruber, we prove that $\tau_{n,1}$ is isomophic to two other representations.

The first description involves binary Lyndon words, and the second involves the action of the symmetric group on the lattice points of the trimmed standard permutahedron.

For the latter one, take all lattice points in the convex hull of all permutations of $(n-2, n-3, \ldots, 1, 0, 0)$, and the natural action.

This proves h-positivity of $\mathrm{Frob}(\tau_{n,1})$.

Conjectures

Conjecture 1

The representation $\tau_{n,1}$ is isomorphic to the representation σ_n.

Conjecture 2

There exists a graded isomorphism of Garsia–Procesi module of the points $\mathrm{Lat}(P_{\delta_n})$ with the module σ_n.

Conjecture 3 (joint work with Markus Reineke)

The coefficient of s_n in the graded Frobenius characteristic of $\sigma_n^{(m-1)}$ equals

$$q^{(n-1)((m-1)n-2)/2}\,\mathrm{DT}_n^{(m)}(1/q).$$