# Friends and Strangers Walking on Graphs

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Princeton University

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This talk is based on joint work with Noga Alon and Colin Defant.

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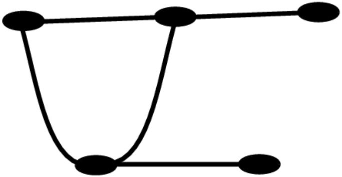

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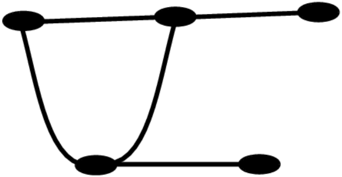

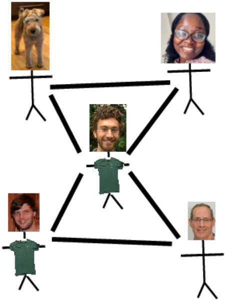

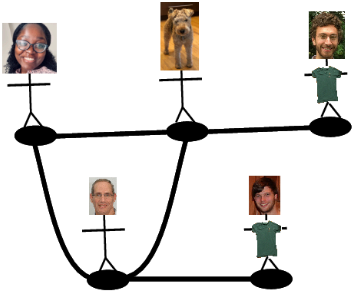

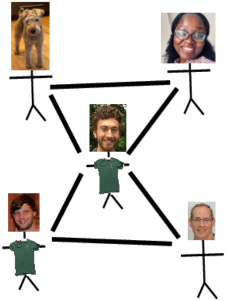

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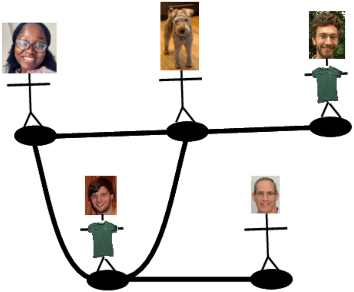

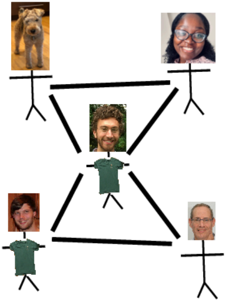

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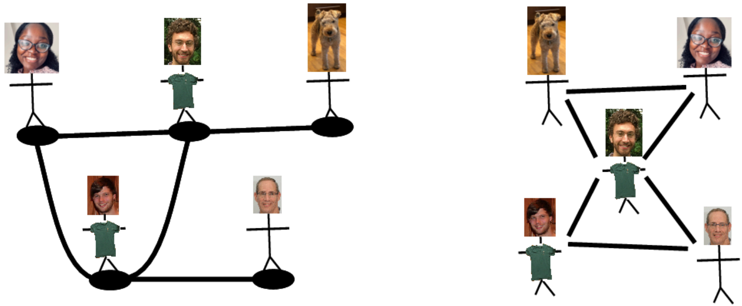

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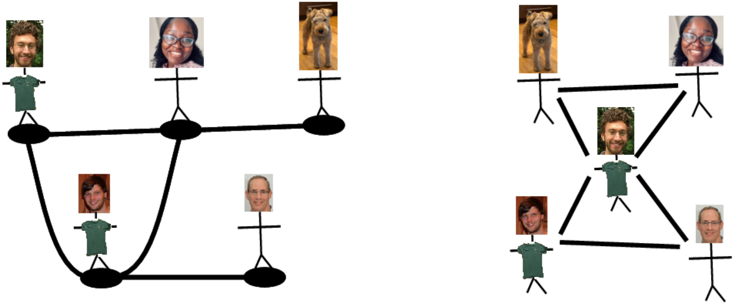

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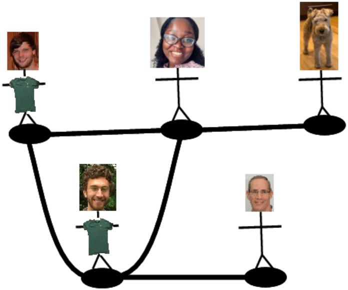

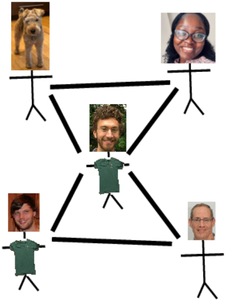

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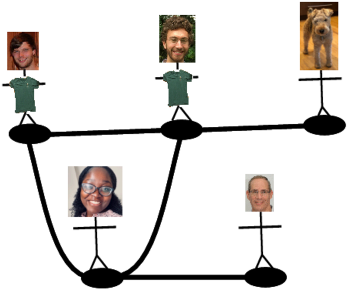

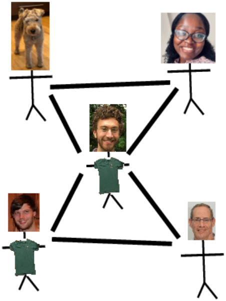

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**Definition (Defant–K.)**

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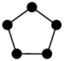
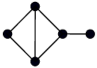
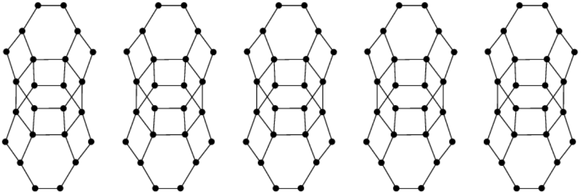
Example: If $X =$  and $Y =$ , then $\mathsf{FS}(X,Y)$ is



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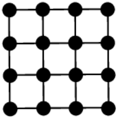
$\mathsf{FS}\Bigg($  ,  $\Bigg)$ .

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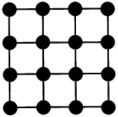
$\mathsf{FS}\Bigg($  ,  $\Bigg)$.

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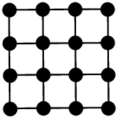
$\mathsf{FS}\Bigg($  ,  $\Bigg)$ .

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Stanley studied the connected components of $\mathsf{FS}(\mathsf{Path}_n, \mathsf{Path}_n)$ in relation to multiplicity-free flag $h$-vectors.

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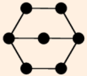
**Theorem (Wilson 1974)**

*Let $X$ be a biconnected non-bipartite graph on $n \geq 3$ vertices that is neither the exceptional graph $\theta_0 =$*  *nor a cycle graph on 4 or more vertices. Then $\mathsf{FS}(X, \mathsf{Star}_n)$ is connected.*

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**Theorem (Defant–K.)**

*There is a bijection $\alpha \mapsto H_\alpha$ from the set of acyclic orientations of $\overline{X}$ to the set of connected components of $\mathsf{FS}(X, \mathsf{Path}_n)$. Vertices in $H_\alpha$ correspond to linear extensions of $\alpha$.*

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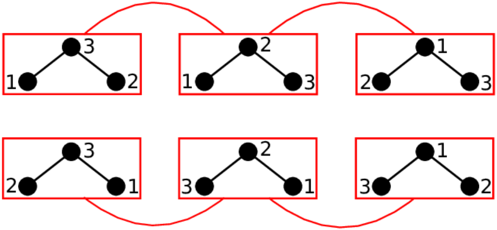
For example, if $X =$  and $\mathsf{Path}_3 =$ 

, then $\mathsf{FS}(X, \mathsf{Path}_3)$ is

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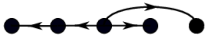

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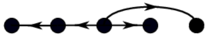

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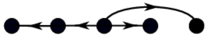

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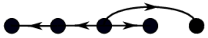

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**Theorem (Defant–K.)**

*There is a bijection* $[\alpha] \mapsto J_{[\alpha]}$ *from the set of double-flip equivalence classes of acyclic orientations of* $\overline{X}$ *to the set of connected components of* $\mathsf{FS}(X, \mathsf{Cycle}_n)$. *Vertices in* $J_{[\alpha]}$ *correspond to linear extensions of* $[\alpha]$.

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**Theorem (Defant–K.)**

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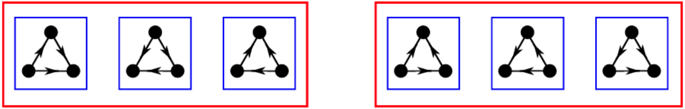

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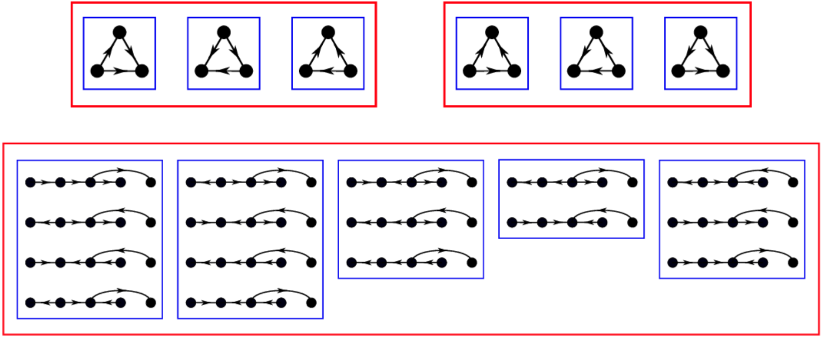

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**Corollary (Defant–K.)**

*The number of connected components of* $\mathsf{FS}(X, \mathsf{Cycle}_n)$ *is* $\nu \cdot T_{\overline{X}}(1,0)$, *where* $\nu$ *is the* gcd *of the sizes of the connected components of* $\overline{X}$ *and* $T_{\overline{X}}$ *denotes the Tutte polynomial of* $\overline{X}$.

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**Corollary (Defant–K.)**

*The graph* $\mathsf{FS}(X, \mathsf{Cycle}_n)$ *is connected if and only if* $\overline{X}$ *is a forest with trees of relatively prime sizes.*

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*We have* $\frac{3}{5}n - 2 \leq d_n \leq \frac{9}{14}n + 1$.

Kiril Bangachev has almost exactly determined the value of $d_n$:

**Theorem (Bangachev)**

*We have* $d_n \leq \left\lceil \frac{3}{5}n \right\rceil$.

In fact, Bangachev also settles many cases of the "asymmetric" version of this problem, where $X$ and $Y$ have different minimum degree requirements.

# Random Graphs

**Theorem (Alon–Defant–K.)**

*Fix any small $\varepsilon > 0$. Let $X$ and $Y$ be independently-chosen random graphs in $\mathcal{G}(n,p)$, where $p = p(n)$ depends on $n$. If*

$$p \leq \frac{2^{-1/2} - \varepsilon}{n^{1/2}},$$

*then* $\mathsf{FS}(X,Y)$ *is disconnected with high probability. If*

$$p \geq \frac{\exp(2(\log n)^{2/3})}{n^{1/2}},$$

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**Theorem (Jeong)**

*The diameters of components of friends-and-strangers graphs can grow super-polynomially in the number of vertices of* $X$ *and* $Y$*. More precisely, for every* $C > 0$*, there exist graphs* $X$ *and* $Y$ *on* $n$ *vertices such that* $\mathsf{FS}(X, Y)$ *has a connected component with diameter greater than* $n^C$.

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Characterize the pairs $(d_1, d_2) \in [0, 1]^2$ such that if $X$ and $Y$ are $n$-vertex graphs with minimum degrees at least $d_1 n + 100$ and $d_2 n + 100$ (respectively), then $\mathsf{FS}(X, Y)$ is guaranteed to be connected.

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**What About... (Alon–Defant–K.)**

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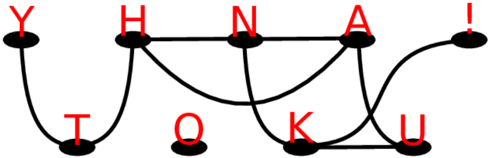
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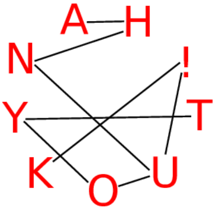
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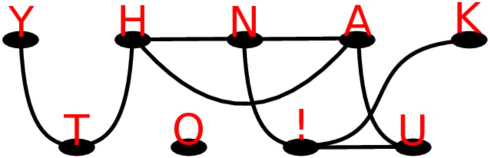
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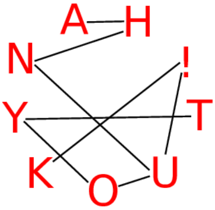
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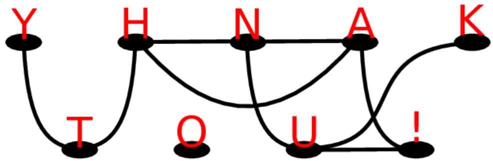
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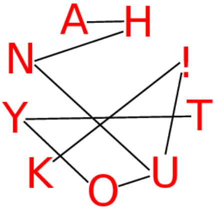
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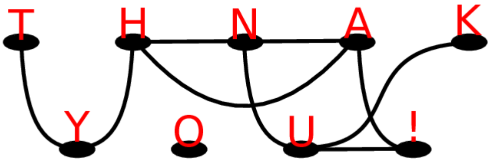
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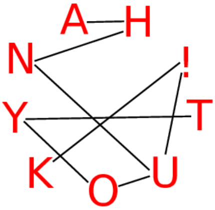
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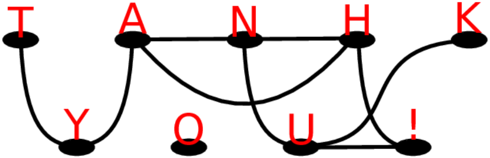
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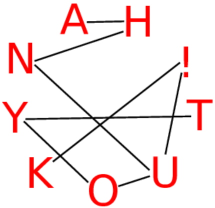
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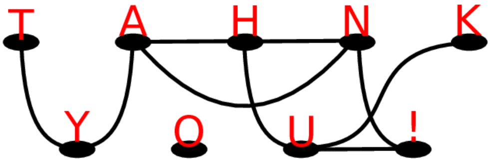
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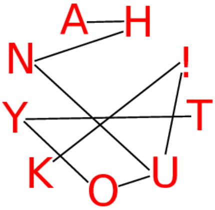
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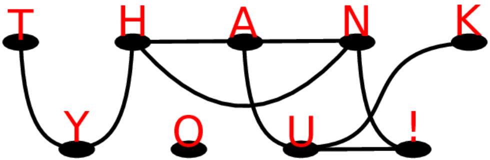
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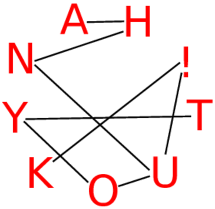
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