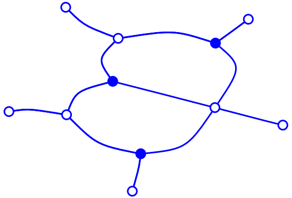
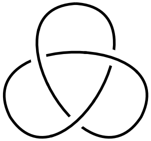
Positroids, knots, and q, t-Catalan numbers

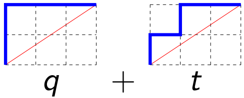
Thomas Lam (U. Michigan)

January, 2022

Joint work with Pavel Galashin (arXiv:2012.09745)







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$$\mathcal{P}(Z; q, t) := \sum_{i,p\in\mathbb{Z}} q^{p-\frac{i}{2}} t^{\frac{d-i}{2}} \dim H^{i,(p,p)}(Z), \quad \text{where } d := \dim Z.$$

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- Point count? Poincaré polynomial? $\mathcal{P}(\Pi^\circ_{k,n}; q, t)$?

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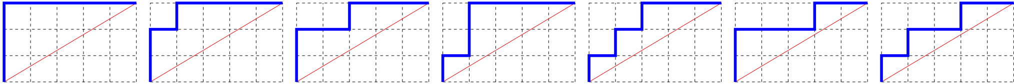
$C_{3,5} = 7$:



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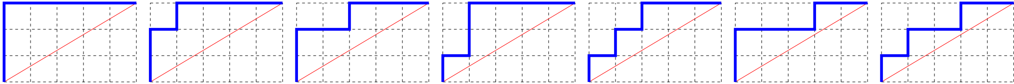
$C_{3,5} = 7$: 

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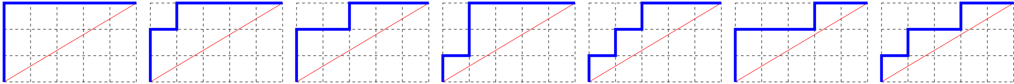

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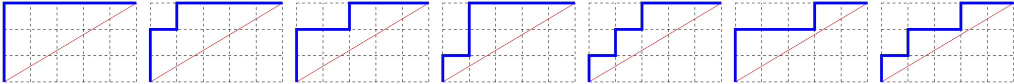
$C_{3,5} = 7$: 

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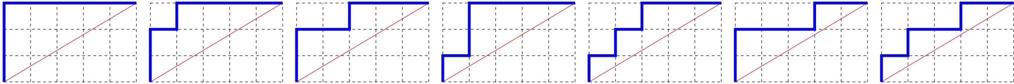
$C_{3,5} = 7$: 

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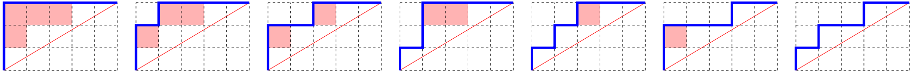

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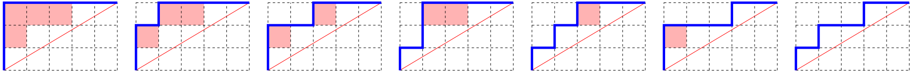

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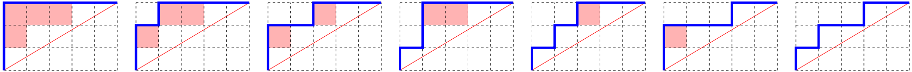

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Theorem (Galashin–L. (2020))

Let $\gcd(k, n) = 1$. *Then the point count and the Poincaré polynomial of* $\Pi^\circ_{k,n}$ *are*

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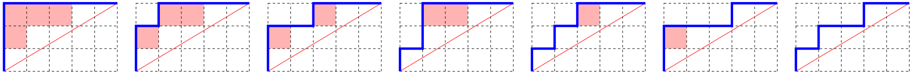

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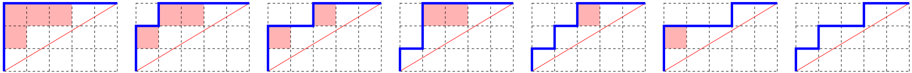

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$\longleftarrow$ does not depend on k?!

Rational q, t-Catalan numbers: (introduced by Garsia–Haiman (1996) and Loehr–Warrington (2009))

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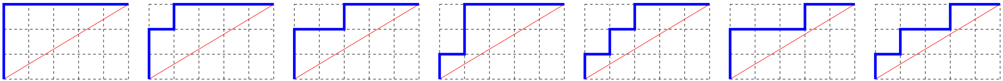

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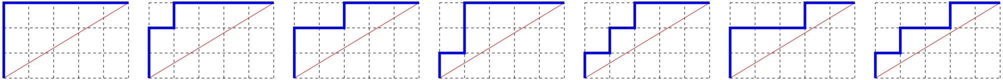

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[Sco06] J. S. Scott. Grassmannians and cluster algebras. *Proc. Lond. Math. Soc. (3)*, 92(2):345–380, 2006.

[GL19] Pavel Galashin and Thomas Lam. Positroid varieties and cluster algebras. *Annales de l'ENS*, to appear.

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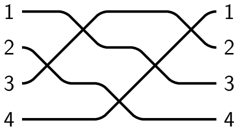

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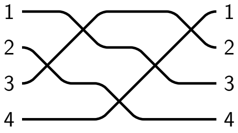

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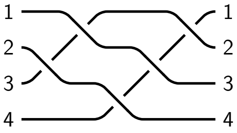

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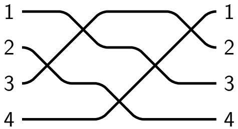

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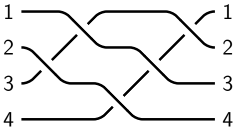

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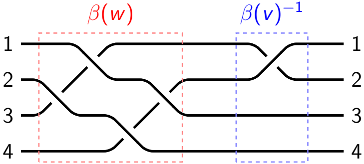

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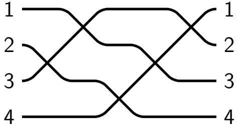

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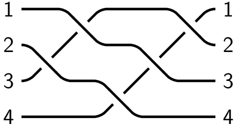

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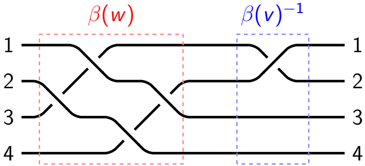

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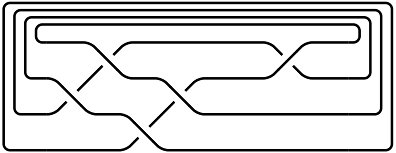

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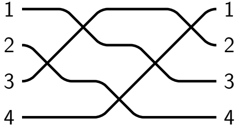

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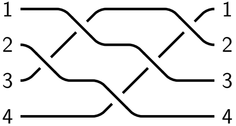

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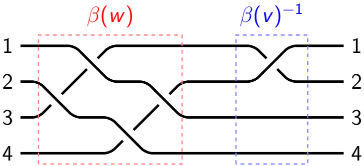

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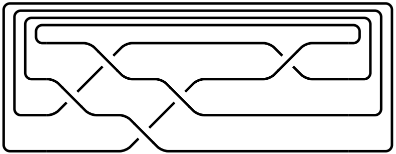

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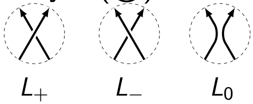

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For $c(wv^{-1}) \geqslant 1$, take the T-equivariant cohomology of $R^\circ_{v,w}$ with compact support instead.

Given a link L, the HOMFLY polynomial $P(L; a, q)$ is defined by $P(\bigcirc) = 1$ and

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Thanks!