A combinatorial Chevalley formula for semi-infinite flag manifolds and its applications

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Geometric interpretation (Schubert calculus on flag manifolds): $s_\lambda(x_1, \ldots, x_k)$ represent Schubert classes σ_λ (i.e., cohomology classes of Schubert varieties) in the cohomology of Grassmannians $Gr_k(\mathbb{C}^n) = SL_n/P_k$:

$$H^*(Gr_k(\mathbb{C}^n)) \simeq Sym(x_1, \ldots, x_k)/I \, .$$

Chevalley formulas (cont.)

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represented by Schubert polynomials $\mathfrak{S}_w \in \mathbb{Z}[x_1, \ldots, x_n]$.

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Chevalley formula:

$$\mathfrak{S}_w \cdot \mathfrak{S}_{s_k} = \sum_{\substack{i \leq k < j \\ \ell(wt_{ij}) = \ell(w)+1}} \mathfrak{S}_{wt_{ij}}\,,$$

where $s_k = t_{k,k+1}$ and $\mathfrak{S}_{s_k} = x_1 + \ldots + x_k$.

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- generalized flag varieties G/B or partial flag varieties G/P (G semisimple Lie group over $\mathbb{C}$, B Borel subgroup, P parabolic subgroup);
- affine versions: affine flag manifold, semi-infinite flag manifold $\mathbf{Q}_G$.

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- $QK_T(G/B)$ is closely related to $K_T(\mathbf{Q}_G)$ (breakthrough of Syu Kato);
- The semi-infinite flag manifolds have applications to the representation theory of affine Lie algebras (level 0 extremal weight modules, Kato-Naito-Sagaki).

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- Chevalley formulas for $QK_T(G/B)$ and $QK_T(G/P)$.
- Applications: more explicit computations and results in type A, for $QK(Fl_n)$.

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Type A_{n-1}:
$P = \{\sum_{i=1}^n \lambda_i \varepsilon_i = (\lambda_1, \ldots, \lambda_n) : \lambda_i \in \mathbb{Z}_{\geq 0}\}$ – compositions;
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Type A_{n-1}: $W = S_n$, $s_i = t_{i,i+1}$, $w_\circ = [n, n-1, \dots, 1]$.

Previous work

$K_T(G/B)$, as module over $K_T(\mathrm{pt}) = \mathbb{Z}[P]$, has a basis of Schubert classes $[\mathcal{O}_{X_w}]$, $w \in W$ (classes of the structure sheaves of Schubert varieties X_w).

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Chevalley formula for $K_T(G/B)$:

$$[\mathcal{L}_\lambda] \cdot [\mathcal{O}_{X_w}] = \sum_{v \in W,\, \mu \in P} c_{w,v}^{\lambda,\mu}\, \mathbf{e}^\mu\, [\mathcal{O}_{X_v}]\,, \qquad c_{w,v}^{\lambda,\mu} \in \mathbb{Z}\,.$$

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[L.-Postnikov]: combinatorial Chevalley formula in terms of the alcove model.

Quantum alcove model: quantum Bruhat graph on the finite Weyl group

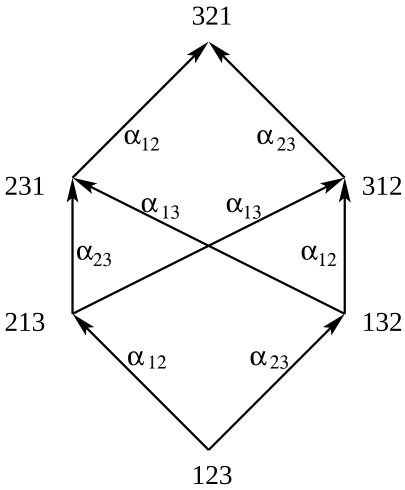
The quantum Bruhat graph on W, denoted $\mathrm{QBG}(W)$, is the directed graph with labeled edges

$$w \stackrel{\alpha}{\longrightarrow} ws_\alpha \,, \qquad \text{where}$$

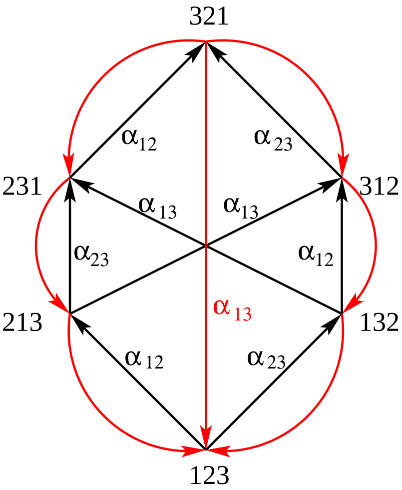
$$\ell(ws_\alpha) = \ell(w) + 1 \quad \text{(covers of Bruhat order)}\,, \qquad \text{or}$$
$$\ell(ws_\alpha) = \ell(w) - 2\mathrm{ht}(\alpha^\vee) + 1\,.$$

(If $\alpha^\vee = \sum_i c_i \alpha_i^\vee$, then $\mathrm{ht}(\alpha^\vee) := \sum_i c_i$.)

Hasse diagram of the Bruhat order for S_3:



Quantum Bruhat graph for S_3:



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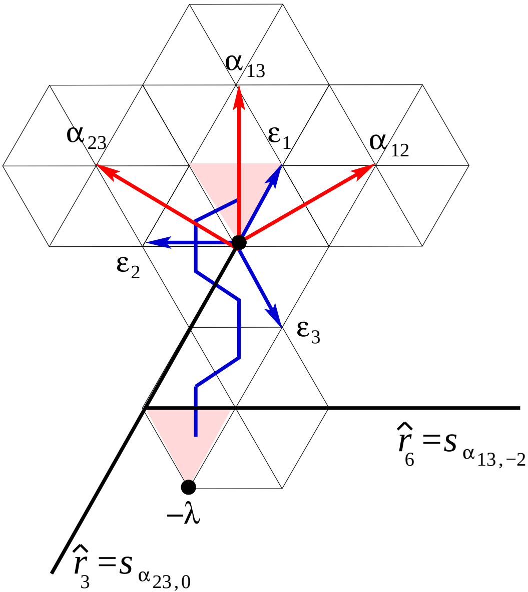
The latter gives a shortest sequence of adjacent alcoves from $A_\circ$ to $A_\circ - \lambda$.

Example. Type A_2, $\lambda = (3,1,0) = 3\varepsilon_1 + \varepsilon_2$,
$\Gamma = (\ (1,2),\ (1,3),\ (2,3),\ (1,3),\ (1,2),\ (1,3)\ )$.



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$$A = \{j_1 < \ldots < j_s\} \subseteq \{1, \ldots, m\}.$$

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$$A = \{j_1 < \ldots < j_s\} \subseteq \{1, \ldots, m\}.$$

For $w \in W$ and A, construct the chain $\pi(w, A)$ of elements in W:

$$w_0 = w, \quad \ldots, \quad w_i := w r_{j_1} \ldots r_{j_i}, \quad \ldots, \quad w_s = \mathrm{end}(w, A).$$

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The main structure structure: w-admissible subsets

$$\mathcal{A}(w, \Gamma) := \{A \,:\, \pi(w, A) \text{ path in } \mathrm{QBG}(W)\}\,.$$

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In addition to $\mathrm{end}(w, A)$, we associate the following statistics with a pair (w, A), for $A = \{j_1 < \ldots < j_s\} \in \mathcal{A}(w, \Gamma)$:

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- $\mathrm{down}(w, A) := \sum_{j \in A^-} |\beta_j|^\vee \in Q^{\vee,+}\,.$

Independence of the quantum alcove model from the λ-chain

Theorem. [Kouno-L.-Naito] Given λ-chains Γ, Γ', there is a sijection [Fisher-Konvalinka] between $\mathcal{A}(w, \Gamma)$ and $\mathcal{A}(w, \Gamma')$ which preserves the relevant statistics.

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Based on quantum Yang-Baxter moves, which are root system analogues of jeu de taquin slides for semistandard Young tableaux (in type A).

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It expresses the product of a Schubert class (indexed by $w \in W_{\mathrm{aff}} = W \ltimes Q^\vee$) with the class of a line bundle corresponding to $\lambda \in P$.

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[L.-Naito-Sagaki]:

- translate the Chevalley formulas for $\lambda \in P^+$ and $\lambda \in P^-$ from quantum LS paths to the quantum alcove model;
- generalize the new formulas to arbitrary $\lambda \in P$, via combinatorics of the quantum alcove model.

Quantum K-theory

Consider variables Q_i for $i \in I$, and let

$$\mathbb{Z}[Q] := \mathbb{Z}[Q_1, \ldots, Q_r], \quad \mathbb{Z}[Q][P] := \mathbb{Z}[Q] \otimes_{\mathbb{Z}} \mathbb{Z}[P].$$

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Given $\xi = d_1\alpha_1^\vee + \cdots + d_r\alpha_r^\vee$ in $Q^{\vee,+}$, let $Q^\xi := Q_1^{d_1} \cdots Q_r^{d_r}$.

The Chevalley formula in $QK_T(G/B)$

Theorem. [L.-Naito-Sagaki, conjecture by L.-Postnikov] Let $k \in I$, and fix a $(-\omega_k)$-chain of roots $\Gamma(-\omega_k)$. Then, in $QK_T(G/B)$, we have the cancellation-free formula:

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$$[\mathcal{O}^{s_k}] \cdot [\mathcal{O}^w] = (1 - \mathbf{e}^{w(\omega_k)-\omega_k})\,[\mathcal{O}^w] + \sum_{A \in \mathcal{A}(w,\Gamma(-\omega_k)) \setminus \{\emptyset\}} (-1)^{|A|-1}\, Q^{\mathrm{down}(w,A)}\, \mathbf{e}^{-\omega_k - \mathrm{wt}(w,A)}\, [\mathcal{O}^{\mathrm{end}(w,A)}]\,.$$

The Chevalley formula in $QK_T(G/B)$

Theorem. [L.-Naito-Sagaki, conjecture by L.-Postnikov] Let $k \in I$, and fix a $(-\omega_k)$-chain of roots $\Gamma(-\omega_k)$. Then, in $QK_T(G/B)$, we have the cancellation-free formula:

$$[\mathcal{O}^{s_k}] \cdot [\mathcal{O}^{w}] = (1 - \mathbf{e}^{w(\omega_k)-\omega_k})\,[\mathcal{O}^{w}] +$$

$$\sum_{A \in \mathcal{A}(w,\Gamma(-\omega_k)) \setminus \{\emptyset\}} (-1)^{|A|-1}\, Q^{\mathrm{down}(w,A)}\, \mathbf{e}^{-\omega_k - \mathrm{wt}(w,A)}\, [\mathcal{O}^{\mathrm{end}(w,A)}]\,.$$

Proof: Translate the corresponding Chevalley formula for the semi-infinite flag manifold via Kato's isomorphism:

$$QK_T(G/B) \xrightarrow{\simeq} K'_T(\mathbf{Q}_G) \subset K_T(\mathbf{Q}_G)\,.$$

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- perform all cancellations via a sign-reversing involution.

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Definition. The quantum Grothendieck polynomial $\mathfrak{G}_w^Q$ is

$$\mathfrak{G}_w^Q := Q(\mathfrak{G}_w) \in \mathbb{Z}[Q, x] \quad \text{for } w \in S_n\,.$$

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Theorem. [L.-Maeno] The quantum Grothendieck polynomials satisfy the version of the above Chevalley formula for $QK(Fl_n)$.

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Theorem. [L.-Naito-Sagaki] In the expansion of $[\mathcal{O}^{s_k}] \cdot [\mathcal{O}^{w}]$, all coefficients are ± 1 (explicitly determined).