The multispecies TAZRP and Macdonald polynomials

Arvind Ayyer, Olya Mandelshtam, and James Martin

Overview:

- modified Macdonald polynomials
- Haglund-Haiman-Loehr tableaux formula
- quinv (queue inversion) statistic
- motivation
- TAZRP (Totally Asymmetric Zero Range Process)
- stationary probabilities of the TAZRP
- sketch of proofs

Macdonald polynomials

- Let $\Lambda \cong \Lambda_{\mathbb{Q}}(q,t)$ be the ring of symmetric polynomials with parameters q,t over $\mathbb{Q}$
- The standard inner product on Λ, with parameters q,t, is $\langle \cdot,\cdot \rangle \cong \langle \cdot,\cdot \rangle_{q,t}$, is defined by: $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda \prod_i \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$
- $P_\lambda(X;q,t)$ is the family of homog. symmetric polynomials in Λ, uniquely determined by Macdonald's triangularity and normalization axioms:
    i. upper triangular with respect to $\{m_\lambda\}$:

$$P_\lambda(X;q,t) = m_\lambda(X) + \sum_{\mu<\lambda} c_{\mu\lambda}(q,t) m_\mu(X)$$

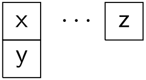
    ii. orthogonal basis for Λ: $\langle P_\lambda, P_\mu \rangle = 0$ if $\lambda \neq \mu$
- modified Macdonald polynomials $\widetilde{H}_\lambda(X;q,t)$ are obtained via plethystic substitution:

$$\widetilde{H}_\lambda(X;q,t) = t^{n(\lambda)} J_\lambda\left[\frac{X}{1-t^{-1}}; q, t^{-1}\right]$$

- they are polynomials with nonnegative integer coefficients in q,t.

Haglund–Haiman–Loehr tableaux formula

- dg(λ) (the diagram of $\lambda = (\lambda_1, \ldots, \lambda_k)$) consists of k bottom justified columns with λ_i boxes, from left to right
- a tableau of type (λ, n) is a filling $\sigma : \mathrm{dg}(\lambda) \to [n]$ of the cells
- inv(σ) is the number of inversions in the configuration



where $x < y < z$ (cyclically mod n)

$\sigma =$

4			
2	2	4	
3	1	1	1
2	3	3	4

$$\lambda = (4, 3, 3, 2)$$

$$x^\sigma = x_1^3 x_2^3 x_3^3 x_4^3$$

$$\mathrm{maj}(\sigma) = 6$$

$$\mathrm{inv}(\sigma) = 1$$

Theorem (Haglund–Haiman–Loehr '05)

$$\widetilde{H}_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\sigma:\mathrm{dg}(\lambda)\to[n]} q^{\mathrm{maj}(\sigma)} t^{\mathrm{inv}(\sigma)} x^\sigma .$$

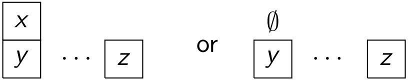
a new statistic: queue-inversion

$\sigma =$

4			
2	2	4	
3	1	1	
2	3	3	4

$\mathrm{quinv}(\sigma) = 4$

- an *L*-triple is a triple of cells in the configuration:

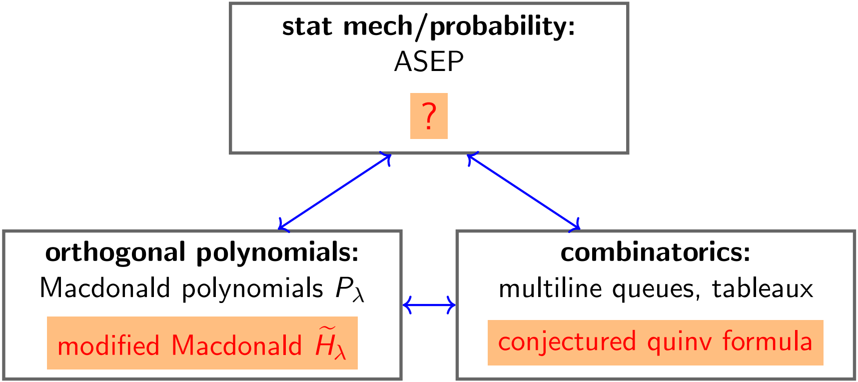


- an *L*-triple forms a quinv (queue-inversion) if $x < y < z$ cyclically mod n (ties are broken by a top-to-bottom and right-to-left reading order)

Theorem (Ayyer–M–Martin '20)

$$\widetilde{H}_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\sigma : \mathrm{dg}(\lambda) \to [n]} q^{\mathrm{maj}(\sigma)} t^{\mathrm{quinv}(\sigma)} x^\sigma.$$

Motivations

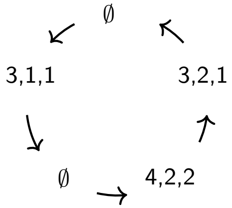


Main goal

Is there an analogous interacting particle system whose probabilities can be described by the same combinatorial objects as $\widetilde{H}_\lambda$?

the multispecies TAZRP

- Fix a (circular 1D) lattice on n sites and a partition λ. Each site of the lattice can have any number of particles
- TAZRP(λ, n) is a Markov chain whose states are *multiset compositions* τ of type λ, with n (possibly empty) parts



$$n = 5$$

$$\lambda = (4,3,3,2,2,1,1,1)$$

$$\tau = (\ \cdot \mid 321 \mid 422 \mid \cdot \mid 311\ )$$

- Each particle is equipped with an exponential clock with parameters $t, x_1, \ldots, x_n$: particles jump counterclockwise to an adjacent site
- The rate of the jump of particle j from site i is

$$f_i(j) = x_i^{-1} t^d \sum_{u=0}^{c-1} t^u,$$

where site i has d particles larger than j and c particles of type j.

E.g.: from $(\ 211 \mid \cdot \mid \cdot)$, the transitions are $\begin{cases} (\ 11 \mid \cdot \mid 2\ ) & \text{with rate } x_1^{-1} \\ (\ 21 \mid \cdot \mid 1\ ) & \text{with rate } x_1^{-1} t(1+t) \end{cases}$

TAZRP probabilities and relation with tableaux

- For a filling σ, read the state $\tau \in \mathsf{TAZRP}(\lambda, n)$ from the bottom row of σ:

$$\tau_j \text{ is the multiset } \{\lambda_i : \sigma(1, i) = j\}$$

- For example, for $\lambda = (2, 1, 1)$ and $n = 3$, the following are all the tableaux that correspond to the state $\big(\, 21 \,\big|\, \cdot \,\big|\, 1 \,\big)$:

1		
1	1	3

2		
1	1	3

3		
1	1	3

1		
1	3	1

2		
1	3	1

3		
1	3	1

Theorem (Ayyer–M–Martin '21)

Fix λ, n. The stationary probability of $\tau \in \mathsf{TAZRP}(\lambda, n)$ is proportional to:

$$\pi(\tau) = \sum_{\substack{\sigma : \mathrm{dg}(\lambda) \to [n] \\ \sigma \text{ has type } \tau}} x^{\sigma} t^{\mathrm{quinv}(\sigma)}$$

Corollary

The so-called partition function of $\mathsf{TAZRP}(\lambda, n)$ is:

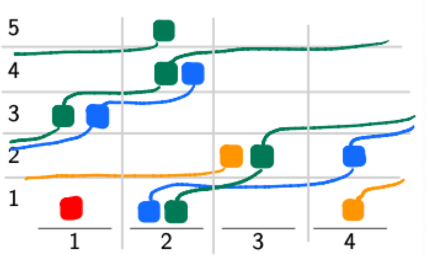
$$\mathcal{Z}_{\lambda,n}(x_1, \ldots, x_n; t) = \widetilde{H}_{\lambda}(x_1, \ldots, x_n; 1, t)$$

tableaux are secretly representing "multiline diagrams"

- the tableaux are representing a queueing system, which is an arrangement of lattice paths arising from particles pairing to the row below, for each pair of rows
- each row of the queueing system a state of the 1-species TAZRP
- in the queueing system, the rows are coupled so as to project onto the bottom row, which is a multispecies TAZRP

2			
2	2		
1	1		
3	4	3	
2	2	4	1

$\longleftrightarrow$



quinv $\longleftrightarrow$ "refusal"

plethystic version of certain non-attacking fillings $\longleftrightarrow$ plethystic version of multiline queues

sketch of proofs: showing $C_\lambda(X;q,t) = \widetilde{H}_\lambda(X;q,t)$

Characterization of $\widetilde{H}_\lambda(X;q,t)$:

i. symmetric in X, and $\langle \widetilde{H}_\lambda(X;q,t), s_{(n)}(X)\rangle = 1$

ii. $\widetilde{H}_\lambda[X(t-1);q,t] = \sum_{\mu\leq\lambda} c_{\mu\lambda}(q,t) m_\mu(X)$

iii. $\widetilde{H}_\lambda[X(q-1);q,t] = \sum_{\mu\leq\lambda'} d_{\mu\lambda}(q,t) m_\mu(X)$

(i) and (ii): same strategy as HHL (LLT polynomials and super-fillings).
(iii): prove the existence of a sign-reversing bijection on super-fillings (using the ordering $1 < 2 < \cdots < \bar{2} < \bar{1}$) that preserves quinv and changes the content by one $\bar{1}$. Our bijection involves:

- the existence of a coinv-preserving bijection ϕ on words that changes the content by one $\bar{1}$

Ex:

w	211	112	121	$21\bar{1}$	$2\bar{1}1$	$12\bar{1}$
$\phi(w)$	$\bar{1}21$	$1\bar{1}2$	$\bar{1}12$	$\bar{1}2\bar{1}$	$\bar{1}\bar{1}2$	$2\bar{1}\bar{1}$
coinv	0	2	1	2	1	3

- quinv-preserving entry-swapping operators τ_j on tableaux

Ex:

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & \bar{1} \\ \hline 3 & \bar{2} & 1 & \bar{1} \\ \hline\end{array} \xrightarrow{\tau_3} \begin{array}{|c|c|c|c|}\hline 1 & 1 & \bar{1} & 2 \\ \hline 3 & \bar{2} & \bar{1} & 1 \\ \hline\end{array} \xrightarrow{\tau_2} \begin{array}{|c|c|c|c|}\hline 1 & \bar{1} & 1 & 2 \\ \hline 3 & \bar{2} & \bar{1} & 1 \\ \hline\end{array} \xrightarrow{\tau_1} \begin{array}{|c|c|c|c|}\hline \bar{1} & 1 & 1 & 2 \\ \hline 3 & \bar{2} & \bar{1} & 1 \\ \hline\end{array}$$