Principal specializations of Schubert polynomials in classical types

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FPSAC 2021

January 19, 2022

Schubert polynomials in type A

- Let $s_i := (i, i+1)$ and $S_n := \langle s_1, s_2, \dots, s_{n-1} \rangle$.
- A *reduced word* for $w \in S_n$ is a word $a_1 a_2 \cdots a_p$ of shortest possible length with $w = s_{a_1} s_{a_2} \cdots s_{a_p}$. Let $\mathrm{Reduced}(w)$ be set of such words.
- A *bounded compatible sequence* for a word $a = a_1 a_2 \cdots a_p$ is a weakly increasing sequence of integers $\mathbf{i} = (i_1 \leq i_2 \leq \cdots \leq i_p)$ with

$$i_j < i_{j+1} \text{ whenever } a_j \leq a_{j+1} \quad \text{and} \quad i_j \leq a_j \text{ whenever } 0 < i_j.$$

  Let $\mathrm{Compatible}(a)$ denote the set of all such sequences.
- The *Schubert polynomial* of $w \in S_n$ has the formula

$$\mathfrak{S}_w := \sum_{a \in \mathrm{Reduced}(w)} \sum_{\substack{\mathbf{i} \in \mathrm{Compatible}(a) \\ 0 < \min(\mathbf{i})}} x_{\mathbf{i}} \in \mathbb{N}[x_1, x_2, \dots, x_{n-1}]$$

  where $x_{\mathbf{i}} := x_{i_1} x_{i_2} \cdots x_{i_p}$ if $\mathbf{i} = (i_1 \leq i_2 \leq \cdots \leq i_p)$.

Macdonald's reduced word formula

There is a remarkable formula for the *principal specialization*

$$\mathfrak{S}_w(1, q, q^2, \dots, q^{n-1})$$

originally conjectured by Macdonald (1991): if $w \in S_n$ then

$$\boxed{\mathfrak{S}_w(1, q, q^2, \dots, q^{n-1}) = \sum_{a = a_1 a_2 \cdots a_p \in \mathrm{Reduced}(w)} \frac{[a_1]_q [a_2]_q \cdots [a_p]_q}{[1]_q [2]_q \cdots [p]_q} q^{\mathrm{comaj}(a)}.}$$

where $\mathrm{comaj}(a) := \sum_{a_i < a_{i+1}} i$ and $[a]_q := \frac{1-q^a}{1-q} = 1 + q + q^2 + \cdots + q^{a-1}$.

- First proof by algebraic argument of Fomin and Stanley (1994)
- Recent bijective proof due to Billey, Holroyd, and Young (2019)

Motivating question: are there analogues for other classical types?

Backstable Schubert polynomials

Taking limits transforms Macdonald's formula into an identity for the *backstable Schubert polynomials* of Lam, Lee, and Shimozono (2018):

$$\overleftarrow{\mathfrak{S}}_w := \sum_{a \in \mathrm{Reduced}(w)} \sum_{\mathbf{i} \in \mathrm{Compatible}(a)} x_{\mathbf{i}} \in \mathbb{N}[[\dots, x_{n-3}, x_{n-2}, x_{n-1}]].$$

Given a power series F, form $F(x_i \mapsto q^{i-1})$ by setting $x_i = q^{i-1}$ $\forall i$. Then

$$\boxed{\overleftarrow{\mathfrak{S}}_w(x_i \mapsto q^{i-1}) = \sum_{a = a_1 a_2 \cdots a_p \in \mathrm{Reduced}(w)} \frac{q^{\mathrm{sum}(a)+\mathrm{comaj}(a)}}{(q-1)(q^2-1)\cdots(q^p-1)}}$$

with $\mathrm{comaj}(a) = \sum_{a_i < a_{i+1}} i$. RHS interpreted as Laurent series in q^{-1}.

This version of Macdonald's formula generalizes to other classical types.

Example for type A

For $w = (1,2)(3,4)$ we have $\text{Reduced}(w) = \{13, 31\}$ and

$$\overleftarrow{\mathfrak{S}}_{(1,2)(3,4)} = \ldots + x_0^2 + 2x_{-1}x_1 + x_{-2}x_2 + x_{-3}x_3 \\ + 2x_0x_1 + x_{-1}x_2 + x_{-2}x_3 \\ + x_1^2 + x_0x_2 + x_{-1}x_3 \\ + x_1x_2 + x_0x_3 \\ + x_1x_3$$

along with $\text{sum}(a) + \text{comaj}(a) = \begin{cases} 5 & a = 13 \\ 4 & a = 31 \end{cases}$ so

$$\boxed{\begin{aligned} \overleftarrow{\mathfrak{S}}_{(1,2)(3,4)}(x_i \mapsto q^{i-1}) &= \cdots + 7q^{-4} + 6q^{-3} + 5q^{-2} + 4q^{-1} + 3 + 2q + q^2 \\ &= \tfrac{q^2}{(1-q^{-1})^2} = \tfrac{q^5}{(q-1)(q^2-1)} + \tfrac{q^4}{(q-1)(q^2-1)}. \end{aligned}}$$

Signed reduced words

- For $0 < i < n$ let $t_i = t_{-i} := (i, i+1)(-i, -i-1)$ and $t_0 := (-1, 1)$.
- Let $\bar{0}$ be a formal symbol with $-1 < \bar{0} < 0 < 1$ and set $t_{\bar{0}} := t_0$.
- Suppose $w \in W_n^{\mathrm{BC}} := \langle t_0, t_1, \dots, t_{n-1} \rangle$.
- A *signed reduced word of type B* for w is a word $a_1 a_2 \cdots a_p$ with

$$a_i \in \{-n+1, \dots, -1, 0, 1, \dots, n-1\}$$

  of shortest possible length such that $w = t_{a_1} t_{a_2} \cdots t_{a_p}$.
- A *signed reduced word of type C* for w is a word $a_1 a_2 \cdots a_p$ with

$$a_i \in \{-n+1, \dots, -1, \bar{0}, 0, 1, \dots, n-1\}$$

  of shortest possible length such that $w = t_{a_1} t_{a_2} \cdots t_{a_p}$.
- Let $\mathrm{Reduced}_B^{\pm}(w)$ and $\mathrm{Reduced}_C^{\pm}(w)$ be these sets of reduced words.

Schubert polynomials in types B/C

The *type B/C Schubert polynomials* of $w \in W_n^{\mathrm{BC}}$ are

$$\mathfrak{S}^{\mathrm{B}}_w := \sum_{\substack{a \in \mathrm{Reduced}^{\pm}_B(w) \\ \mathbf{i} \in \mathrm{Compatible}(a)}} x_{\mathbf{i}} \quad \text{and} \quad \mathfrak{S}^{\mathrm{C}}_w := \sum_{\substack{a \in \mathrm{Reduced}^{\pm}_C(w) \\ \mathbf{i} \in \mathrm{Compatible}(a)}} x_{\mathbf{i}} = 2^{\ell_0(w)} \mathfrak{S}^{\mathrm{B}}_w$$

where $\ell_0(w) := |\{ i \in \mathbb{Z} : w(i) < 0 < i \}|$.

Introduced by Billey–Haiman, both $\mathfrak{S}^{\mathrm{B}}_w$ and $\mathfrak{S}^{\mathrm{C}}_w$ are already "backstable."

Let $\mathrm{Reduced}_C(w)$ be set of words in $\mathrm{Reduced}^{\pm}_C(w)$ with all letters ≥ 0.

Theorem (M.–Pawlowski (2020))

$$\mathfrak{S}^{C}_w(x_i \mapsto q^{i-1}) = \sum_{a = a_1 a_2 \cdots a_p \in \mathrm{Reduced}_C(w)} \frac{(q^{a_1}+1)(q^{a_2}+1)\cdots(q^{a_p}+1)}{(q-1)(q^2-1)\cdots(q^p-1)} q^{\mathrm{comaj}(a)}.$$

Example for types B/C

If $w = (1,-2)(2,-1) \in W_n^{\mathsf{BC}}$ then $\mathrm{Reduced}_C(w) = \{010\}$ and

$$\mathrm{Reduced}^{\pm}_C(w) = \{010,\ \bar{0}10,\ 0\bar{1}0,\ 01\bar{0},\ \bar{0}\bar{1}0,\ \bar{0}1\bar{0},\ 0\bar{1}\bar{0},\ \bar{0}\bar{1}\bar{0}\}.$$

One has $\mathrm{comaj}(010) = 1$ and one can compute that

$$\begin{aligned}\mathfrak{S}^{\mathsf{C}}_{(1,-2)(2,-1)} = \cdots &+ 4x_{-2}x_{-1}^2 + 4x_{-2}^2x_0 + 8x_{-3}x_{-1}x_0 + 4x_{-4}x_0^2 \\ &+ 4x_{-3}x_0^2 + 8x_{-2}x_{-1}x_0 \\ &+ 4x_{-2}x_0^2 + 4x_{-1}^2x_0 \\ &+ 4x_{-1}x_0^2\end{aligned}$$

$$\Rightarrow \quad \boxed{\begin{aligned}\mathfrak{S}^{\mathsf{C}}_{(1,-2)(2,-1)}(x_i \mapsto q^{i-1}) &= \cdots + 20q^{-7} + 12q^{-6} + 8q^{-5} + 4q^{-4} \\ &= \tfrac{4q^{-4}}{(1-q^{-1})^2(1-q^{-3})} = \tfrac{(q^0+1)(q^1+1)(q^0+1)}{(q-1)(q^2-1)(q^3-1)}q.\end{aligned}}$$

Even signed reduced words

- For $1 < i < n$, let $r_i = r_{-i} := (i, i+1)(-i, -i-1) = t_i$ but define

$$r_1 := (1,2)(-1,-2) = t_1 \quad \text{and} \quad r_{-1} := (1,-2)(-1,2) = t_0 t_1 t_0.$$

- Suppose $w \in W_n^{\mathrm{D}} := \langle r_{-1}, r_1, r_2, \dots, r_{n-1} \rangle$.
- A *signed reduced word of type D* for w is a word $a_1 a_2 \cdots a_p$ with

$$a_i \in \{-n+1, \dots, -2, -1, 1, 2, \dots, n-1\}$$

of shortest possible length such that $w = r_{a_1} r_{a_2} \cdots r_{a_p}$.
- Let $\mathrm{Reduced}^{\pm}_D(w)$ be the set of such words.

Schubert polynomials in type D

Defined by Billey–Haiman, the *type D Schubert polynomial* of $w \in W_n^{\mathrm{D}}$ is

$$\mathfrak{S}^{\mathrm{D}}_w = \sum_{\substack{a \in \mathrm{Reduced}^{\pm}_D(w) \\ \mathbf{i} \in \mathrm{Compatible}(a)}} x_{\mathbf{i}} \in \mathbb{Z}[[\dots, x_{-1}, x_0, x_1, \dots, x_{n-1}]].$$

Suppose $a = a_1 a_2 \cdots a_p$ where $a_i \in \{\pm 1, \pm 2, \pm 3, \dots, \pm(n-1)\}$. Define

$$\mathrm{comaj}_{\mathrm{D}}(a) := |\{i : a_i > 0\}| + 2 \textstyle\sum_{a_i \prec a_{i+1}} i$$

where $\prec$ is the order $-1 \prec -2 \prec \cdots \prec -n \prec 1 \prec 2 \prec \cdots \prec n$.

If $a = a_1 a_2 a_3 a_4 = \bar{1}\bar{2}31$ then $\mathrm{comaj}_{\mathrm{D}}(a) = 2 + 2(1+2) = 8$.

Theorem (M.–Pawlowski (2020))

$$\mathfrak{S}^{D}_w(x_i \mapsto q^{i-1}) = \textstyle\sum_{a \in \mathrm{Reduced}^{\pm}_D(w)} \frac{(q^{|a_1|+1})(q^{|a_2|}+1)\cdots(q^{|a_p|}+1)}{(q^2-1)(q^4-1)\cdots(q^{2p}-1)} q^{\mathrm{comaj}_{\mathrm{D}}(a)}.$$

Example for type D

If $w = (1,-1)(4,-4) \in W_n^{\mathrm{D}}$ then $\mathrm{Reduced}_D^{\pm}(w)$ has 32 elements, formed by adding signs to the letters in $a_1a_2a_3a_4a_5a_6 = 321123$ in all ways that give opposite signs to the two entries with absolute value one. One has

$$\begin{aligned}\mathfrak{S}^{\mathrm{D}}_{(1,-1)(4,-4)} = \cdots &+ x_0^4x_1x_3 + 2x_{-1}x_0^3x_2x_3 + 2x_{-1}^2x_0x_1x_2x_3 + 2x_{-2}x_0^2x_1x_2x_3 \\ &+ x_0^4x_2x_3 + 2x_{-1}x_0^2x_1x_2x_3 \\ &+ x_0^3x_1x_2x_3\end{aligned}$$

$$\Rightarrow \quad \boxed{\begin{aligned}&\mathfrak{S}^{\mathrm{D}}_{(1,-1)(4,-4)}(x_i \mapsto q^{i-1}) \\ &\qquad = \cdots + 27q^{-4} + 15q^{-3} + 7q^{-2} + 3q^{-1} + 1 \\ &\qquad = \frac{(1+q^{-2})}{(1-q^{-1})^3(1-q^{-3})^2(1-q^{-5})}\end{aligned}}$$

which coincides with $\sum_{a\in\mathrm{Reduced}_D^{\pm}(w)} \frac{(q^{|a_1|+1})(q^{|a_2|}+1)\cdots(q^{|a_p|}+1)}{(q^2-1)(q^4-1)\cdots(q^{2p}-1)} q^{\mathrm{comaj}_{\mathrm{D}}(a)}$.