FPSAC21 poster: Specializations of colored quasisymmetric functions

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Motivation

For a permutation $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$, let

- $\mathrm{Des}(w) := \{1 \le i \le n-1 : w_i > w_{i+1}\}$
- $\mathrm{des}(w) := \#\,\mathrm{Des}(w)$
- $\mathrm{maj}(w) := \sum_{i \in \mathrm{Des}(w)} i$

be the descent set, descent number and major index of w, respectively.

Theorem (Euler, MacMahon, Carlitz and others)

For a positive integer n,

$$\sum_{m \ge 0} [m+1]_q^n x^m = \frac{\sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)}}{(1-x)(1-qx)\cdots(1-q^n x)}$$

$$\sum_{w \in \mathfrak{S}_n} q^{\mathrm{maj}(w)} = [n]_q!,$$

where $[n]_q := 1 + q + \cdots + q^{n-1}$ and $[n]_q! := [1]_q [2]_q \cdots [n]_q$.

The quasisymmetric generating function associated to $\mathfrak{S}_n$ is known to satisfy

$$F(\mathfrak{S}_n; \mathbf{x}) := \sum_{w \in \mathfrak{S}_n} F_{n, \mathrm{Des}(w)}(\mathbf{x}) = (x_1 + x_2 + \cdots)^n,$$

where $F_{n,S}(\mathbf{x})$ is the fundamental quasisymmetric function associated to $S \subseteq [n-1]$. A proof would follow by taking the

- stable principal specialization defined by the substitutions

$$x_1 = 1,\ x_2 = q,\ x_3 = q^2,\ \ldots$$

- principal specialization of order m defined by the substitutions

$$x_1 = 1,\ x_2 = q,\ \ldots,\ x_m = q^{m-1},\ x_{m+1} = x_{m+2} = \cdots = 0.$$

Colored permutation statistics

The r-colored permutation group $\mathfrak{S}_{n,r}$ consists of all permutations $\sigma : \Omega_{n,r} \to \Omega_{n,r}$ of

$$\Omega_{n,r} := \{1^{r-1} <_c \cdots <_c n^{r-1} <_c \cdots <_c 1^1 <_c \cdots <_c n^1 <_c 1^0 <_c \cdots <_c n^0\}$$

such that $\sigma(i^0) = j^\beta \Rightarrow \sigma(i^\alpha) = j^{\alpha+\beta}$. The descent set $\mathrm{Des}(w^\epsilon)$ of an r-colored permutation $w^\epsilon = w_1^{\epsilon_1} w_2^{\epsilon_2} \cdots w_n^{\epsilon_n} \in \mathfrak{S}_{n,r}$ is the set of all

- $1 \le i \le n-1$ such that $w_i^{\epsilon_i} >_c w_{i+1}^{\epsilon_{i+1}}$
- $i = 0$, whenever $\epsilon_1 \ne 0$.

Let $\mathrm{des}(w^\epsilon) := \#\,\mathrm{Des}(w^\epsilon), \mathrm{Des}^*(w^\epsilon) := \mathrm{Des}(w^\epsilon) \setminus \{0\},\ \mathrm{des}^*(w) = \#\,\mathrm{Des}^*(w^\epsilon)$. Also,let

- $\mathrm{csum}(w^\epsilon) := \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$
- $\mathrm{maj}(w^\epsilon) := \sum_{i \in \mathrm{Des}^*(w^\epsilon)} i$
- $\mathrm{fmaj}(w^\epsilon) := r\,\mathrm{maj}(w^\epsilon) + \mathrm{csum}(w^\epsilon)$
- $\mathrm{fdes}(w^\epsilon) := r\,\mathrm{des}^*(w^\epsilon) + \epsilon_1$.

be the color sum, major index, flag major index and flag descent number of w^ϵ, respectively.

Colored quasisymmetric functions

Let $\mathbf{x}^{(j)} = (x_1^{(j)}, x_2^{(j)}, \ldots)$ be sequences of commuting indeterminates for all $0 \le j \le r-1$. The fundamental colored quasisymmetric function associated to $w^\epsilon \in \mathfrak{S}_{n,r}$ is the following formal power series in $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(r-1)}$

$$F_{w^\epsilon}(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(r-1)}) := \sum_{\substack{i_1 \ge i_2 \ge \cdots \ge i_n \\ j \in \mathrm{Des}^*(w^\epsilon) \Rightarrow i_j > i_{j+1}}} x_{i_1}^{(\epsilon_1)} x_{i_2}^{(\epsilon_2)} \cdots x_{i_n}^{(\epsilon_n)}$$

Specializations; General formulas

For $\mathcal{A} \subseteq \mathfrak{S}_{n,r}$ consider the colored quasisymmetric generating function

$$F(\mathcal{A}) = F(\mathcal{A}; \mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(r-1)}) = \sum_{w \in \mathcal{A}} F_w(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(r-1)}).$$

Let $\mathrm{ps}_{q,p}^{(r)}$ be the specialization defined by the substitutions

- $x_i^{(j)} = q^{i-1} p^j$, for all $i \ge 1$ and $0 \le j \le r-1$.

Let $\mathrm{ps}_{q,p,m}^{(r)}$ be the specialization defined by the substitutions

- $x_i^{(0)} = q^{i-1}$, for all $1 \le i \le m$,
- $x_i^{(j)} = q^{i-1} p^j$, for all $1 \le i \le m-1$ and $1 \le j \le r-1$,
- $x_i^{(j)} = 0$, otherwise.

Theorem

For all $\mathcal{A} \in \mathfrak{S}_{n,r}$,

$$\sum_{m \ge 1} \mathrm{ps}_{q,p,m}^{(r)} F(\mathcal{A})\, x^{m-1} = \frac{\sum_{w \in \mathcal{A}} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)} p^{\mathrm{csum}(w)}}{(1-x)(1-qx)\cdots(1-q^n x)}$$

$$\mathrm{ps}_{q,p}^{(r)} F(\mathcal{A}) = \frac{\sum_{w \in \mathcal{A}} q^{\mathrm{maj}(w)} p^{\mathrm{csum}(w)}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

Setting $q \to q^r$ and $p \to pq$, yields formulas for fmaj instead of maj.

Let $\phi_{q,p}^{(r)}$ be the specialization defined as follows: If $m = rs + t$ for some $1 \le t \le r$ and $s \ge 0$, then

- $x_i^{(j)} = 0$, if $(i,j) >_{\mathrm{lex}} (rs+1, t-1)$
- $x_i^{(j)} = \begin{cases} q^{i+j-1} p_j, & \text{if } i \equiv 1 \pmod r \\ 0, & \text{if } i \not\equiv 1 \pmod r \end{cases}$, otherwise.

Example

For $r = 3$ and $m \in \{7, 8, 9\}$, specialization $\phi_{q,p,m}^{(r)}$ becomes

$$(x_i^{(j)})_{\substack{0 \le j \le 2 \\ i \ge 1}} = \begin{cases} \begin{pmatrix} 1 & 0 & 0 & q^3 & 0 & 0 & q^6 & 0 & \cdots \\ qp & 0 & 0 & q^4 p & 0 & 0 & 0 & 0 & \cdots \\ q^2p^2 & 0 & 0 & q^5p^2 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix}, & \text{if } m = 7 = 2 \cdot 3 + 1 \\ \begin{pmatrix} 1 & 0 & 0 & q^3 & 0 & 0 & q^6 & 0 & \cdots \\ qp & 0 & 0 & q^4 p & 0 & 0 & q^7 p & 0 & \cdots \\ q^2p^2 & 0 & 0 & q^5p^2 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix}, & \text{if } m = 8 = 2 \cdot 3 + 2 \\ \begin{pmatrix} 1 & 0 & 0 & q^3 & 0 & 0 & q^6 & 0 & \cdots \\ qp & 0 & 0 & q^4 p & 0 & 0 & q^7 p & 0 & \cdots \\ q^2p^2 & 0 & 0 & q^5p^2 & 0 & 0 & q^8p^2 & 0 & \cdots \end{pmatrix}, & \text{if } m = 9 = 2 \cdot 3 + 3. \end{cases}$$

Theorem

For all $\mathcal{A} \subseteq \mathfrak{S}_{n,r}$,

$$\sum_{m \ge 1} \phi_{q,p,m}^{(r)} F(\mathcal{A})\, x^{m-1} = \frac{\sum_{w \in \mathcal{A}} x^{\mathrm{fdes}(w)} q^{\mathrm{fmaj}(w)} p^{\mathrm{csum}(w)}}{(1-x)(1-q^r x^r)(1-q^{2r}x^r)\cdots(1-q^{nr}x^r)}.$$

Applications; Euler–Mahonian idendities

Let

- $\mathcal{D}_{n,r}$ be the set of all r-colored permutations without fixed points of zero color, called colored derangements
- $\mathcal{I}_{n,r}$ be the set of all r-colored permutations which equal their own conjugate-inverse, that is all $w^\epsilon \in \mathfrak{S}_{n,r}$ such that $w^\epsilon = (w^{-\epsilon})^{-1}$, called absolute involutions.

We have

$$F(\mathfrak{S}_{n,r}) = \left(h_1(\mathbf{x}^{(0)}) + \cdots + h_1(\mathbf{x}^{(r-1)})\right)^n$$

$$F(\mathcal{D}_{n,r}) = \sum_{k=0}^{n} (-1)^k e_k(\mathbf{x}^{(0)}) \left(h_1(\mathbf{x}^{(0)}) + \cdots + h_1(\mathbf{x}^{(r-1)})\right)^{n-k}$$

$$\sum_{n \ge 0} F(\mathcal{I}_{n,r}) z^n = \prod_{t=0}^{r-1} \prod_{i \ge 1} (1 - z x_i^{(t)})^{-1} \prod_{1 \le i < j} (1 - z^2 x_i^{(t)} x_j^{(t)})^{-1},$$

where e_n (resp. h_n) is the n-th elementary (resp. complete homogeneous) symmetric function.

Applying the specializations of the second column to these formulas yields refined Euler–Mahonian identities on $\mathfrak{S}_{n,r}, \mathcal{D}_{n,r}, \mathcal{I}_{n.r}$.

Corollaries (Euler–Mahonian identities)

For a nonnegative integer m, we write $m = r\mathrm{Q}(m) + \mathrm{R}(m)$ for some nonnegative integer $\mathrm{Q}(m)$ and $0 \le \mathrm{R}(m) < r$. Then

$$\sum_{m \ge 0} \left([m+1]_q + p[r-1]_p [m]_q\right)^n x^m = \frac{\sum_{w \in \mathfrak{S}_{n,r}} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)} p^{\mathrm{csum}(w)}}{(1-x)(1-qx)\cdots(1-q^n x)}$$

$$\sum_{w \in \mathfrak{S}_{n,r}} q^{\mathrm{maj}(w)} p^{\mathrm{csum}(w)} = [r]_p^n [n]_q!$$

$$\sum_{m \ge 0} \left([m+1]_{q^r} + pq[m]_{q^r}[r-1]_{pq}\right)^n x^m = \frac{\sum_{w \in \mathfrak{S}_{n,r}} x^{\mathrm{des}(w)} q^{\mathrm{fmaj}(w)} p^{\mathrm{csum}(w)}}{(1-x)(1-q^r x)\cdots(1-q^{nr} x)}$$

$$\sum_{w \in \mathfrak{S}_{n,r}} q^{\mathrm{fmaj}(w)} p^{\mathrm{csum}(w)} = [r]_{qp}^n [n]_{q^r}!$$

$$\sum_{m \ge 0} \left([\mathrm{Q}(m)+1]_{q^r} + pq[r-1]_{pq}[\mathrm{Q}(m)]_{q^r} + pq^{r\mathrm{Q}(m)+1}[\mathrm{R}(m)]_{pq}\right)^n x^m$$

$$= \frac{\sum_{w \in \mathfrak{S}_{n,r}} x^{\mathrm{fdes}(w)} q^{\mathrm{fmaj}(w)} p^{\mathrm{csum}(w)}}{(1-x)(1-q^r x^r)(1-q^{2r}x^r)\cdots(1-q^{nr}x^r)}$$

$$\sum_{m \ge 0} \sum_{k=0}^{n} (-1)^k q^{r\binom{k}{2}} \binom{m+1}{k}_{q^r} \left([m+1]_{q^r} + pq[m]_{q^r}[r-1]_{pq}\right)^{n-k} x^m$$

$$= \frac{\sum_{w \in \mathcal{D}_{n,r}} x^{\mathrm{des}(w)} q^{\mathrm{fmaj}(w)} p^{\mathrm{csum}(w)}}{(1-x)(1-q^r x)\cdots(1-q^{nr} x)}$$

$$\sum_{w \in \mathcal{D}_{n,r}} q^{\mathrm{fmaj}(w)} p^{\mathrm{csum}(w)} = [r]_{pq}^n [n]_{q^r}! \sum_{k=0}^{n} (-1)^k \frac{q^{r\binom{k}{2}}}{[r]_{pq}^k [k]_{q^r}!}$$

$$\sum_{n \ge 0} \frac{\sum_{w \in \mathcal{I}_{n,r}} x^{\mathrm{des}(w)} p^{\mathrm{csum}(w)}}{(1-x)^{n+1}} z^n = \sum_{m \ge 0} \frac{x^m}{(1+z)^m (1-z)^{m+1} (z;p)_r^m (z^2;p^2)_r^{\binom{m}{2}}}.$$

Although these identities appear in the work of various authors, the sixth identity seems to be new even in the case $r = 1$.

Bottomline

"nice formula" for $F(\mathcal{A})$ + "nice specialization" = Euler–Mahonian identity on $\mathcal{A}$