The saturation problem for refined Littlewood-Richardson coefficients

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Littlewood-Richardson (LR) coefficients

The Schur polynomials S_λ for $\lambda \in \mathcal{P}[n]$ form a basis of $\mathbb{C}[\mathbf{x}]^{S_n}$ in n-variables $\mathbf{x} = (x_1, x_2, ..., x_n)$.
$\mathcal{P}[n] = \{\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n) | \lambda_i \in \mathbb{Z}_{\geq 0}, 1 \leq i \leq n\}$.

$$S_\lambda S_\mu = \sum_{\nu \in P[n]} c^\nu_{\lambda,\mu} S_\nu$$

The constants $c^\nu_{\lambda,\mu}$ is called the Littlewood-Richardson coefficients.

Demazure operators on $\mathbb{C}[\mathbf{x}]$

$$T_i(\mathbf{x}) := \frac{x_i f(x_1, x_2, ..., x_n) - x_{i+1} f(x_1, .., x_{i-1}, x_{i+1}, x_i, x_{i+2}, .., x_n)}{x_i - x_{i+1}}$$

for $1 \leq i \leq n-1$.

Refined Littlewood-Richardson coefficient

For $w \in S_n$, let $T_w = T_{i_1} T_{i_2} \dots T_{i_k}$, where $w = s_{i_1} s_{i_2} \dots s_{i_k}$ is any reduced word of w.

Key polynomial

For $w \in S_n$ and $\mu \in P[n]$, the *key polynomial* is $\chi_{w,\mu}(\mathbf{x}) := T_w(\mathbf{x}^\mu)$, where $\mathbf{x}^\mu = \prod_{i=1}^n x_i^{\mu_i}$. Note that $T_{w_0} : \mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{x}]^{S_n}$ where w_0 is the longest element in S_n.

w-refined Littlewood-Richardson coefficient

For $w \in S_n$ and $\lambda, \mu \in P[n]$

$$T_{w_0}(\mathbf{x}^\lambda \cdot \chi_{w,\mu}(\mathbf{x})) = \sum_{\nu \in P[n]} c^\nu_{\lambda,\mu}(w) S_\nu.$$

The coefficients $c^\nu_{\lambda,\mu}(w)$ is called the w-refined Littlewood-Richardson coefficients.

Kostant-Kumar (KK) module

Properties

(1): $c^{\nu}_{\lambda,\mu}(1) = \delta_{\lambda+\mu,\nu}$ (2): $c^{\nu}_{\lambda,\mu}(w) \leq c^{\nu}_{\lambda,\mu}(w')$ if $w \leq w'$
(3): $c^{\nu}_{\lambda,\mu}(w) = c^{\nu}_{\mu,\lambda}(w^{-1})$ (4): $c^{\nu}_{\lambda,\mu}(w_0) = c^{\nu}_{\lambda,\mu}$

Kostant-Kumar module

Given $\lambda, \mu \in P[n]$ and $w \in S_n$, the cyclic $GL_n(\mathbb{C})$-submodule of $V(\lambda) \otimes V(\mu)$ generated by $v_\lambda \otimes v_{w\mu}$ is called a Kostant-Kumar module and denoted by $K(\lambda, w, \mu)$, where v_λ is the highest weight vector of $V(\lambda)$ and $v_{w\mu}$ is a non zero weight vector of weight $w\mu$ of $V(\mu)$.

Kumar 1988

$\mathbf{Char}K(\lambda, w, \mu) = T_{w_0}(\mathbf{x}^{\lambda} \cdot \chi_{w,\mu}(\mathbf{x})) \implies c^{\nu}_{\lambda,\mu}(w)$ is the multiplicity of $V(\nu)$ in $K(\lambda, w, \mu)$.

Saturation problem

Saturation property

A permutation $w \in S_n$ is said to have the *saturation property* if the following hold for all $\lambda, \mu, \nu \in P[n]$:

$$c_{k\lambda,k\mu}^{k\nu}(w) > 0 \text{ for some } k \geq 1 \implies c_{\lambda,\mu}^{\nu}(w) > 0.$$

Both permutations $w = 1$ and $w = w_0$ have the saturation property, established by Knutson-Tao (1999).

312-avoiding permutation

A permutation w contains 312-pattern if there exist $i < j < k$ such that $w(j) < w(k) < w(i)$. A permutation dose not containing any 312-pattern is called 312-*avoiding permutation*. A permutation dose not containing any 231-pattern is called 231-*avoiding permutation*.

Main problem

Theorem[__, Raghavan, Viswanath (2021)]

1. Let $w \in S_n$ be either 312-avoiding or 231-avoiding. Then w has the saturation property.
2. More generally, let $H = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_p} \subseteq S_n$ be a Young subgroup and $w = w_1 w_2 \cdots w_p \in H$ such that each $w_i \in S_{n_i}$ is either 312-avoiding or 231-avoiding. Then w has the saturation property.

Note that if w is 312-avoiding then w^{-1} is 231-avoiding and

$$c^{\nu}_{\lambda\mu}(w) = c^{\nu}_{\mu,\lambda}(w^{-1}).$$

Gelfand-Tsetlin (GT) pattern

Let $\mu = (\mu_1, \mu_2, ..., \mu_n) \in \mathcal{P}[n]$.

Gelfand-Tsetlin (GT) pattern

$$\begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & & a_{22} & & & \\ & & a_{31} & & a_{32} & & a_{33} & & \\ & a_{41} & & a_{42} & & a_{43} & & a_{44} & \\ a_{51} & & a_{52} & & a_{53} & & a_{54} & & a_{55} \end{array} \qquad \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 3 & & 2 & & & \\ & & 5 & & 3 & & 1 & & \\ & 5 & & 4 & & 2 & & 1 & \\ 6 & & 4 & & 2 & & 2 & & 1 \end{array}$$

Where $a_{ni} = \mu_i$ and $NE_{ij} = a_{ij} - a_{i-1,j} \geq 0$, $SE_{ij} = a_{i-1,j} - a_{i,j+1} \geq 0$ for all $i > j$. .

The GT polytope of shape μ is denoted by $\mathrm{GT}(\mu)$ and the integral points in $\mathrm{GT}(\mu)$ is denoted by $\mathrm{GT}_{\mathbb{Z}}(\mu)$

Kogan face of GT polytope

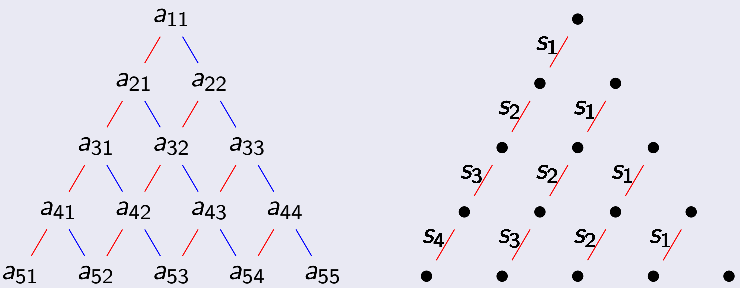
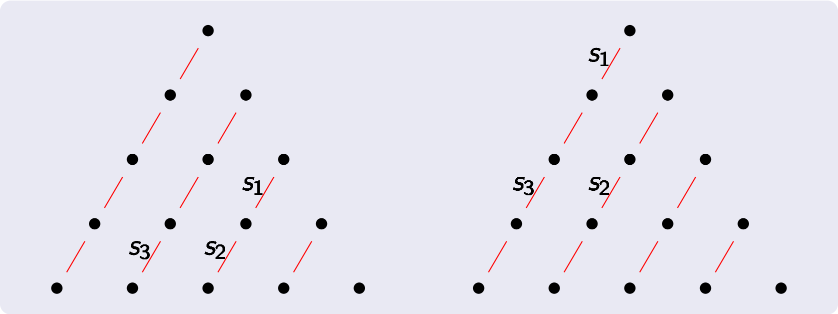


Figure: Gelfand-Tsetlin array for $n = 5$. The red edges $a_{ij} \longrightarrow a_{i-1,j}$ are labelled by s_{i-j} for $i > j$.

Fix a subset $F \subseteq \{(i,j) : n \geq i > j \geq 1\}$.



$\sigma(F)$ is the product of edge labels NE_{ij} for $(i,j) \in F$ in the lexicographic order.

For example let $F_1 = \{(4,3),(5,2),(5,3)\}$ and $F_2 = \{(2,1),(4,1),(4,2)\}$ then:

$$\sigma(F_1) = s_1 s_3 s_2 = \sigma(F_2).$$

Kogan face

If $\operatorname{len}\sigma(F) = |F|$, we say that F is *reduced*.

Kogan face $\mathrm{K}(\mu, F) = \{A \in \mathrm{GT}(\mu) | NE_{ij} = 0, \forall (i,j) \in F \text{ in } A\}$.

Fujita 2020

$$\text{Set: } \varpi(F) = w_0\, \sigma(F)\, w_0,$$

For $w \in S_n$, define

$$\mathrm{K}(\mu, w) := \cup\, \mathrm{K}(\mu, F),$$

the union is taken over all reduced F for which $\varpi(F) = w$.

Hive

Given $\lambda, \mu, \nu \in \mathcal{P}[n]$ such that $|\lambda| + |\mu| = |\nu|$

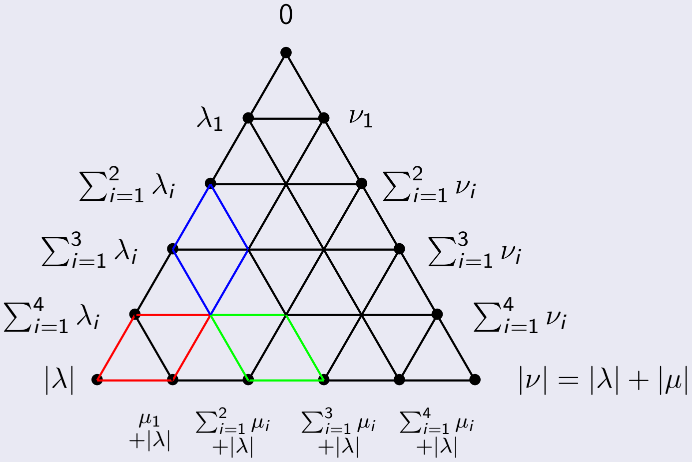
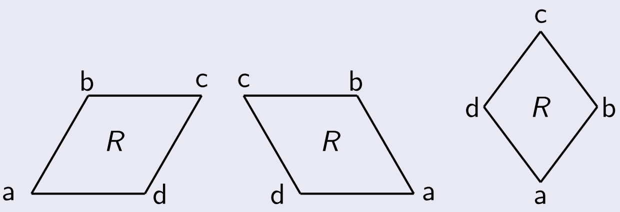


Figure: A hive in Hive(λ, μ, ν) for $n = 5$.

Rhombus inequalities

Consider the following rhombus R:



Then content$(R) = (b + d) - (a + c) \geq 0$.
It is well known that $|\text{Hive}_{\mathbb{Z}}(\lambda, \mu, \nu)| = c_{\lambda\mu}^{\nu}$ (and is a variation of proofs in [Buch], [Pak, Vallejo]).

Map from Hive to GT

For given λ, μ, ν, the hive polytope denoted by Hive(λ, μ, ν). There exist a injective linear map $\boldsymbol{\partial}$: Hive$(\lambda, \mu, \nu) \rightarrow$ GT(μ).

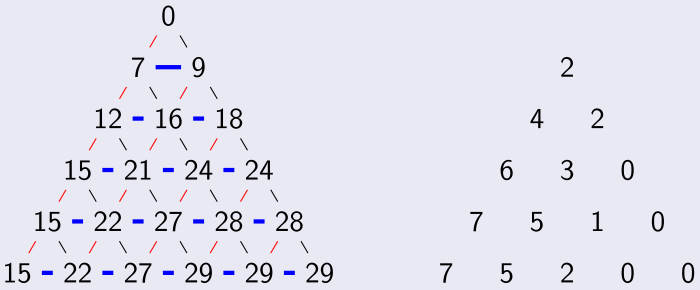
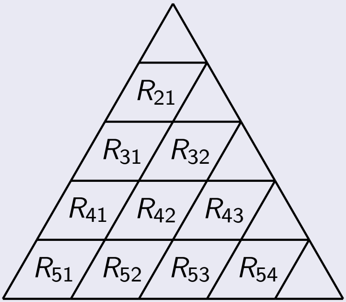
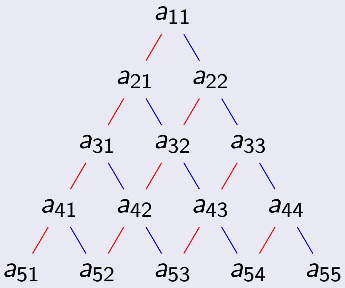


Figure: The hive on the left maps under $\boldsymbol{\partial}$ to the GT pattern on the right (example borrowed from King et. al.)





$NE_{ij}(= a_{ij} - a_{(i-1)j})$ in $\partial h = \text{content}(R_{ij})$ in h for $n \geq i > j \geq 1$.

A rhombus with zero content is said to be *flat* rhombus.

Hive Kogan face

Given $F \subset \{(i,j) : n \geq i > j \geq 1\}$, recall that $\mathrm{K}(\mu, F)$ is the face of $\mathrm{GT}(\mu)$ on which $NE_{ij} = 0 \ \forall (i,j) \in F$.

The Hive kogan face $\mathrm{K}^{\mathrm{Hive}}(\lambda, \mu, \nu, F) := \boldsymbol{\partial}^{-1}\,\mathrm{K}(\mu, F)$

$$= \{h \in \mathrm{Hive}(\lambda, \mu, \nu) : R_{ij} \text{ is flat in } h \text{ for all } (i,j) \in F\}.$$

we say it *reduced* if F is reduced. For $w \in S_n$, define:

$$\mathrm{K}^{\mathrm{Hive}}(\lambda, \mu, \nu, w) := \boldsymbol{\partial}^{-1}(\mathrm{K}(\mu, w))$$

Theorem [due to Lakshmibai-Littelmann-Magyar, Joseph, Fujita, Buch.]

$c_{\lambda\mu}^{\nu}(w) = \#\,\mathrm{K}_{\mathbb{Z}}^{\mathrm{Hive}}(\lambda, \mu, \nu, w_0 w).$

Increasable subset of hive

Increasable subset of hive [Knutson-Tao and Buch]

A subset S of the interior vertices of hive h is said to be *increasable*, if there is an $\epsilon > 0$ such that $h' = h + \epsilon I_S \in \mathrm{Hive}(\lambda, \mu, \nu)$.

Proposition [Knutson-Tao]

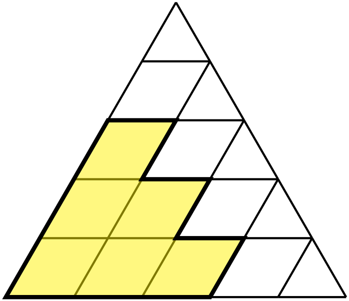
Let $\lambda, \mu, \nu \in \mathcal{P}[n]$ be regular (i.e., $\lambda_i \neq \lambda_j$ if $i \neq j$, and likewise for μ, ν) with $|\lambda| + |\mu| = |\nu|$. Let h satisfy the following properties:

1. h is a vertex of the hive polytope $\mathrm{Hive}(\lambda, \mu, \nu)$.
2. h has no increasable subsets.

Then each interior label of h is an integral linear combination of its boundary labels. In particular $h \in \mathrm{Hive}_{\mathbb{Z}}(\lambda, \mu, \nu)$.

Hive Kogan face for 312-avoiding permutation

Let $w \in S_n$ be 312-avoiding $\implies$ $w_0 w$ is 132-avoiding and there exists a unique reduced $F_w \subset \{(i,j) : n \geq i > j \geq 1\}$ such that $\varpi(F_w) = w_0 w$. Further, it has the following form



$\mathrm{K}^{\mathrm{Hive}}(\lambda, \mu, \nu, w_0 w) = \mathrm{K}^{\mathrm{Hive}}(\lambda, \mu, \nu, F_w)$ on which the R_{ij} are flat for all $(i,j) \in F_w$.

Increasable subset for hive in 312-avoiding hive Kogan face

Let w be 312-avoiding and let F_w be as above.

Lemma

Let $h \in \mathrm{K}^{\mathrm{Hive}}(\lambda, \mu, \nu, F_w)$ and let S be an increasable subset for h, say $h' = h + \epsilon I_S \in \mathrm{Hive}(\lambda, \mu, \nu)$ for some $\epsilon > 0$. Then $h' \in \mathrm{K}^{\mathrm{Hive}}(\lambda, \mu, \nu, F_w)$.

Consider the polyhedral cone $\mathrm{K}^{\mathrm{Hive}}(-, w_0 w)$ is the set of all $\mathbb{R}$-labellings of vertices of hive triangle subject to:

1. content$(R) \geq 0$ for all rhombi R in hive.
2. content$(R_{ij}) = 0$ for all $(i, j) \in F_w$.

Consider the projection $\pi : \mathrm{K}^{\mathrm{Hive}}(-, w_0 w) \to (\mathbb{R}^n)^3$ defined by $\pi(h) = (\lambda, \mu, \nu)$. Denote the image of π by Horn$(w_0 w)$.

Proof of main theorem

The saturation property is equivalent to:

$$\mathrm{K}^{\mathrm{Hive}}_{\mathbb{Z}}(\,-\,, w_0 w) \cap \pi^{-1}(\lambda,\mu,\nu) \neq \emptyset, \forall (\lambda,\mu,\nu) \in \mathrm{Horn}_{\mathbb{Z}}(w_0 w).$$

This follows from $c^{\nu}_{\lambda\mu}(w) = \#\,\mathrm{K}^{\mathrm{Hive}}_{\mathbb{Z}}(\lambda,\mu,\nu,w_0 w)$ and $\mathrm{K}^{\mathrm{Hive}}(p\lambda, p\mu, p\nu, w_0 w) = p\,\mathrm{K}^{\mathrm{Hive}}(\lambda,\mu,\nu,w_0 w), \forall p > 0$ in $\mathbb{R}$.

Knutson-Tao and Buch

Choose $\zeta : \mathrm{K}^{\mathrm{Hive}}(\,-\,, w_0 w) \to \mathbb{R}$, $\zeta(h) =$ a generic positive linear combination of its vertex labels. Then,

1. $\forall (\lambda,\mu,\nu) \in \mathrm{Horn}(w_0 w)$, the maximum value of ζ on $\pi^{-1}(\lambda,\mu,\nu)$ attains at a unique point say $h_{(\lambda,\mu,\nu)}$.
2. The map $\ell : \mathrm{Horn}(w_0 w) \to \mathrm{K}^{\mathrm{Hive}}(\,-\,, w_0 w)$, $(\lambda,\mu,\nu) \mapsto h_{(\lambda,\mu,\nu)}$, is continuous and piecewise-linear.

Proof of main theorem

1. $\ell(\lambda, \mu, \nu)$ is a vertex of $\mathrm{K}^{\mathrm{Hive}}(\lambda, \mu, \nu, w_0 w)$.
2. $h = \ell(\lambda, \mu, \nu)$ has no increasable subsets.

Then for λ, μ, ν regular, each label of $\ell(\lambda, \mu, \nu)$ is an integer linear combination of the λ_i, μ_i, ν_i, $1 \leq i \leq n$.
Following Knutson-Tao and Buch, by the continuity of ℓ, it follows that each piece of ℓ is a linear function of $(\lambda, \mu, \nu) \in \mathbb{R}^{3n}$ with $\mathbb{Z}$-coefficients. As a corollary:

$$\ell(\mathrm{Horn}_{\mathbb{Z}}(w_0 w)) \subseteq \mathrm{K}_{\mathbb{Z}}^{\mathrm{Hive}}(\lambda, \mu, \nu, w_0 w)$$

This proves the saturation property for 312-avoiding permutations.

Thank You