Inset games and strategies

Kento NAKADA[1]

January 18, 2022

[1]nakada@okayama-u.ac.jp. This paper is partially supported by KAKENHI 18H01435.

Since the beginning of combinatorial game theory, i.e. the 1930's [7][3], winning strategies have been studied. In this paper, we focus on impartial games [2], which are a sort of 2-player games with perfect information. Winning strategies of impartial games are analyzed by the Sprague-Grundy value. Since the Sprague-Grundy value is recursively defined, it is very difficult to calculate the Sprague-Grundy value for most games. Therefore, finding large classes for which it is possible to calculate the Sprague-Grundy value is a very important issue. Typical examples of games possible to calculate the Sprague-Grundy value by "good" algorithm are:

- Nim. See [1][2] for details.
- Sato-Welter game. This game is introduced by M. Sato [6] and C. P. Welter [8] independently. See section 3 for definition and properties.
- Turning turtles. See section 4 for definition and properties. See [1] for further details.

In this paper, we focus on the class of insets, which is one of the classes of d-complete posets. We define a game over insets — which we call the *inset games* — and a closed formula for their Sprague-Grundy function. Our games are costructed as patchworks of the Sato-Welter games.
This paper is organized as follows: Section 2 explains fundamental notions and definitions of games and the Sprague-Grundy values. Section 3 explains precise definition of the Sato-Welter games. Section 4 explains precise definition of the turning turtles. Section 5 explains our main results.

We begin by defining the games we concern with.

Definition

Let P be a set, and $\to$ a binary relation over P. For an element $p \in P$, we put $\alpha(p) := \{q \in P \mid p \to q\}$. The pair $(P; \to)$ is called a game if it satisfies:

1. *For any $p \in P$, the set $\alpha(p)$ is finite;*
2. *There exists no infinite sequence*

$$p_0, p_1, p_2, p_3, \ldots \quad p_i \in P$$

with

$$p_i \to p_{i+1}, \quad i = 0, 1, 2, 3, \ldots$$

We call an element p of P a *position*, $\alpha(p)$ the *option set* at the position p. If $\alpha(p) = \varnothing$, then we say p is an *ending position*. Any position $p = p_0 \in P$ can be interpreted as an opening position of a 2-player game (in the usual sense of the word); two players alternatively choose positions:

$$\begin{array}{ll} p_0 \to p_1 & \text{(the first player's move)}, \\ p_1 \to p_2 & \text{(the second player's move)}, \\ p_2 \to p_3 & \text{(the first player's move)}, \\ & \cdots\cdots \end{array}$$

until one of them reaches an ending position p_n. If n is odd (resp. even), we say the first (resp. second) player *wins*. If $(P; \to)$ and $(Q; \to)$ are isomorphic to each other as digraphs, then we say $(P; \to)$ is game-isomorphic to $(Q; \to)$.

Example (1-heap nim)

Denote by $\mathbb{N}$ *the set of nonnegative integers. Then, the pair* $(\mathbb{N}; >)$ *is a game, where* $>$ *denotes the ordinary order relation 'greater than'. This game is called the* 1*-heap nim.*

According to [7][3], we define:

Definition

For a game $(P; \to)$*, let* $\mathrm{SG} = \mathrm{SG}_P : P \to \mathbb{N}$ *be the map defined by*

$$\mathrm{SG}(p) = \min\left(\mathbb{N} \setminus \left\{\mathrm{SG}(q) \in \mathbb{N} \middle| p \to q\right\}\right), \qquad (p \in P).$$

The map SG *is called the Sprague-Grundy function of* P*. The value* $\mathrm{SG}(p)$ *is called the Sprague-Grundy number (or Sprague-Grundy value) of* p*.*

Example (1-heap nim)

The Sprague-Grundy function SG *of the* 1*-heap nim* $(\mathbb{N}; >)$ *is the identity map:*

$$\mathrm{SG}(x) = x \ \ (x \in \mathbb{N}).$$

Propsition ([7][3])

Let $(P; \to)$ *be a game and* $p \in P$. *Then we have:*

1. *If* $\mathrm{SG}_P(p) = 0$, *then, for any* $q \in \alpha(p)$, *we have* $\mathrm{SG}_P(q) > 0$.
2. *If* $\mathrm{SG}_P(p) > 0$, *then, for some* $q \in \alpha(p)$, *we have* $\mathrm{SG}_P(q) = 0$.
3. *If* p *is an ending position, then we have* $\mathrm{SG}_P(p) = 0$.

Remark

For a position $p \in P$, the following two conditions are equivalent:

1. *the position p has a winning strategy.*
2. $\mathrm{SG}_P(p) > 0$.

Indeed, if the first player is at the position p with $\mathrm{SG}_P(p) > 0$. Then there exists a next position $q \in \alpha(p)$ with $\mathrm{SG}_P(q) = 0$. The strategic move $p \to q$ leads the first player to win, because any next position $r \in \alpha(q)$ chosen by the second player must be satisfying $\mathrm{SG}_P(r) > 0$.

Denote by $\mathbb{Z}$ the set of integers. We shall write the binary expression of an integer $a \in \mathbb{Z}$ as

$$a = [a_i] = [a_i]_{i \in \mathbb{N}} = [\dots, a_i, \dots, a_3, a_2, a_1, a_0].$$

For example, we have:

$$\begin{aligned} 11 &= 1 + 2 + 0 + 2^3 + 0 + 0 + \cdots = [\cdots 001011], \\ -1 &= 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + \cdots = [\cdots 111111], \\ -12 &= 0 + 0 + 2^2 + 0 + 2^4 + 2^5 + \cdots = [\cdots 110100]. \end{aligned}$$

We recall the definition of Nim-addition $\oplus$ in $\mathbb{Z}$. For $a = [a_i]$, $b = [b_i]$, and $c = [c_i]$ in $\mathbb{Z}$, we write

$$a \oplus b = c$$

if

$$a_i + b_i \equiv c_i \pmod 2, \qquad i \in \mathbb{N}.$$

For example, we have

$$3 \oplus 5 = [\cdots 00011] \oplus [\cdots 00101] = [\cdots 00110] = 6.$$

The system $(\mathbb{Z}; \oplus)$ forms an abelian group with

$$a \oplus a = 0, \quad \text{for any } a \in \mathbb{Z}.$$

Note that $\mathbb{N}$ is an index 2 subgroup of $(\mathbb{Z}; \oplus)$. We have

$$(-1) \oplus a = -a - 1.$$

Here, the symbol $-$ denotes the inverse on the usual addition (the binary operation $+$).

For $a \in \mathbb{Z}$, we put

$$\mathrm{N}(a) = a \oplus (a-1).$$

Lemma

Let $a, b, c \in \mathbb{Z}$. We have:

1. *If a is a multiple of 2^t $(t \in \mathbb{N})$, and is not a multiple of 2^{t+1}, then*

$$\mathrm{N}(a) = [\cdots 0\overbrace{11\cdots 1}^{t+1}] = 2^{t+1} - 1.$$

2. $\mathrm{N}(a)$ *is negative if and only if $a = 0$.*

We also put, for $a, b \in \mathbb{Z}$,

$$\left(a \middle| b\right) = \mathrm{N}(a \oplus b).$$

We have the following ([2, ch.13], [6], [8]).

Lemma

Let $a, b, c \in \mathbb{Z}$. We have:

1. *If $\left(a \middle| x\right) = \left(b \middle| x\right)$ for any $x \in \mathbb{Z}$, then $a = b$.*
2. $\left(a \middle| b\right) = \mathrm{N}(a - b)$.
3. $\left(a \middle| b\right) = \left(a + c \middle| b + c\right) = \left(a \oplus c \middle| b \oplus c\right)$.
4. *If $c > 0$ (resp. $c < 0$), then we have*

$$a \oplus \sum_{h=0}^{c-1}{}^{\oplus} \left(a \middle| h\right) = a - c \quad \left(\textit{resp. } a \oplus \sum_{h=c}^{-1}{}^{\oplus} \left(a \middle| h\right) = a - c \right).$$

Following Conway [2], we call a function $f : \mathbb{Z} \to \mathbb{Z}$ of the form

$$f(x) = \cdots ((((x \oplus a) + b) \oplus c) + d) \oplus \cdots$$

an *animating function*. Clearly an animating function is a bijective map from $\mathbb{Z}$ to $\mathbb{Z}$, and its inverse is animating again. As is shown in [2], a function f is animating if and only if it can be written as

$$f(x) = x \oplus \sum_{i=1}^{r}{}^{\oplus} \left(x \middle| \alpha_i\right) \oplus \beta \tag{1}$$

with some $\alpha_i, \beta \in \mathbb{Z}$. Moreover, the expression (2) is unique as long as $\alpha_1, \alpha_2, \ldots, \alpha_r$ are distinct. We denote by $\mathrm{Anim}(\mathbb{Z})$ the set of animating functions.

Lemma (Sato [6] and Conway [2])

1. *The set* $\mathrm{Anim}(\mathbb{Z})$ *is closed under the composition and the inverse. (Hence* $(\mathrm{Anim}(\mathbb{Z}), \circ)$ *forms a group.)*
2. *If* f *is an element of* $\mathrm{Anim}(\mathbb{Z})$, *then we have*
$$(f(x) \,|\, f(y)) = (x \,|\, y), \quad (x, y \in \mathbb{Z}).$$
3. *If* $y = f(x)$ *with* $f(x)$ *given by*
$$f(x) = x \oplus \sum_{i=1}^{r}{}^{\oplus} (x \,|\, \alpha_i) \oplus \beta \tag{2}$$
then the inverse $x = f^{-1}(y)$ *is given by*
$$f^{-1}(y) = y \oplus \sum_{i=1}^{r}{}^{\oplus} (y \,|\, f(\alpha_i)) \oplus \beta.$$

Definition

A multivariate function $E : \mathbb{Z}^n \to \mathbb{Z}$ *is said to be animating if*

1. $E_i(x_i) := E(x_1, \cdots, x_i, \cdots, x_n)$ *is animating for each* x_i,
2. E *is symmetric in* $x_1, \cdots, x_n$.

The set of animating functions over $\mathbb{Z}^n$ *is denoted by* $\mathrm{Anim}(\mathbb{Z}^n)$.

Definition

Fix an integer $n \geq 1$. *Put*

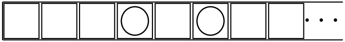
$$P_n := \left\{ \mathbf{x} \subseteq \mathbb{N} \,\middle|\, |\mathbf{x}| = n \right\}.$$

For $\mathbf{x}, \mathbf{y} \in P_n$, *we denote* $\mathbf{x} \to_{\mathrm{A}} \mathbf{y}$ *if the following two coditions hold:*

- $|\mathbf{x} \cap \mathbf{y}| = n - 1$;
- *if* $x \in \mathbf{x} \setminus \mathbf{y}$ *and* $y \in \mathbf{y} \setminus \mathbf{x}$, *then* $y < x$.

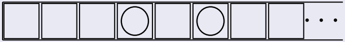
We call the game $(P_n; \to_{\mathrm{A}})$ *the Sato-Welter game with* n *balls.*

The Sato-Welter game with n balls can be visually interpreted as follows: n balls are lined up. At each move, a player moves one ball "◯" leftwards to any empty box. The player to make the last move wins.

$\mathbf{x} = \{3, 5\} =$  $\in P_2$

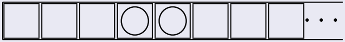
Example

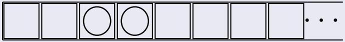
For a position

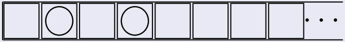
$\mathbf{x} = \{3,5\} =$  $\in P_2$

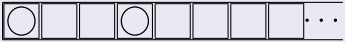
of the Sato-Welter game of 2 *balls,*

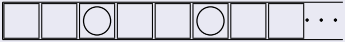
the elements of the option set $\alpha(\mathbf{x})$ *are:*

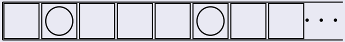
$\{3,4\} =$ 

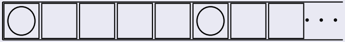
$\{3,2\} =$ 

$\{3,1\} =$ 

$\{3,0\} =$ 

$\{2,5\} =$ 

$\{1,5\} =$ 

$\{0,5\} =$ 

Remark

By regarding a position $\mathbf{x} \in P_n$ *of the Sato-Welter game as a beta number, we can also regard* $\mathbf{x}$ *as Young diagrams. For exmaple, a position* $\mathbf{x} = \{3, 5\} \in P_2$ *is regarded as a Young diagram with partition* $(4, 3)$*.*

For $\mathbf{x} = \{x_1, x_2, \cdots, x_n\} \in P_n$, we put

$$\varphi_n(\mathbf{x}) := \sum_i^{\oplus} x_i \oplus \sum_{i<j}^{\oplus} (x_i \,|\, x_j).$$

For $\mathbf{x} \in P_n$, we have $\varphi_n(\mathbf{x}) \geq 0$. Since φ is a symmetric function in $x_1, \cdots, x_n$, we denote $\varphi_n(\mathbf{x}) = \varphi_n(x_1, x_2, \cdots, x_n)$.

Theorem (Sato [6], and Welter [8])

The Sprague-Grundy function of the Sato-Welter game $(P_n; \to_{\mathrm{A}})$ is given by

$$\varphi_n(\mathbf{x}), \quad \mathbf{x} \in P_n.$$

Example

For $\mathbf{x} = \{3, 5\} \in P_2$, *the Sprague-Grundy value* $\varphi_2(\mathbf{x})$ *is*

$$\varphi_2(\mathbf{x}) = 3 \oplus 5 \oplus \left(3 \middle| 5\right) = 5 \neq 0.$$

Hence, the position $\mathbf{x}$ *has a winning strategy. The (unique) winning move is:*

$$\{3, 5\} \rightarrow_{\mathrm{A}} \{3, 2\}.$$

Definition

Put

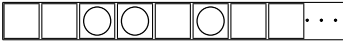
$$P_{odd} := \bigcup_{n:odd} P_n = \bigcup_{n:odd} \left\{ \mathbf{x} \subseteq \mathbb{N} \,\middle|\, |\mathbf{x}| = n \right\}. \tag{3}$$

For $\mathbf{x} \in P_n$ *and* $\mathbf{y} \in P_m$, *we denote* $\mathbf{x} \to_{\mathrm{D}} \mathbf{y}$ *if*

- $n = m \text{ and } \mathbf{x} \to_{\mathrm{A}} \mathbf{y}$; *or*
- $n = m + 2 \text{ and } \mathbf{x} \supset \mathbf{y}$.

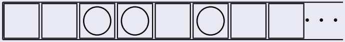
We call the game $(P_{odd}; \to_{\mathrm{D}})$ *the turning turtles.*

The turning turtles can be visually interpreted as follows: Several turtles are lined up. An odd number of them are awake and the others sleeping. At each move, a player chooses two turtles with both hands and turns them over. The player is allowed to turn "◯"(awake turtle) into "empty"(sleeping turtle) and also "empty" into "◯". However, the turtle that he grabs with his right hand must be an awake turtle. The player to make the last move wins.

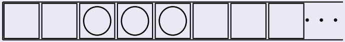
$\mathbf{x} = \{2, 3, 5\} =$  $\in P_{\mathrm{odd}}$

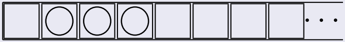
Example

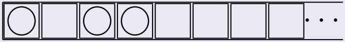
For a position

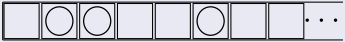
$\mathbf{x} = \{2, 3, 5\} =$  $\in P_{odd}$

of the turning turtles, the elements of the option set $\alpha(\mathbf{x})$ *are:*

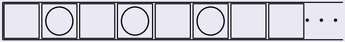
$\{2, 3, 4\} =$ 

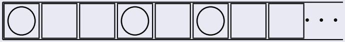
$\{2, 3, 1\} =$ 

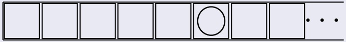
$\{2, 3, 0\} =$ 

$\{2, 1, 5\} =$ 

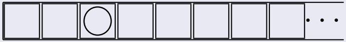
$\{2, 0, 5\} =$ 

$\{1, 3, 5\} =$ 

$\{0, 3, 5\} =$ 

$\{5\} =$ 

$\{3\} =$ 

$\{2\} =$ 

Remark

By regarding a position $\mathbf{x} \in P_{odd}$ *of the turning tutles as a strict partition, we can also regard* $\mathbf{x}$ *as shifted Young diagrams. For exmaple, a position* $\mathbf{x} = \{2, 3, 5\} \in P_{odd}$ *is regarded as a shifted Young diagram with strict partition* $(5, 3, 2)$.

For $\mathbf{x} = \{x_1, x_2, \cdots, x_n\} \in P_{\mathsf{odd}}$, we put

$$\psi(\mathbf{x}) := x_1 \oplus x_2 \oplus \cdots \oplus x_n.$$

For $\mathbf{x} \in P_{\mathsf{odd}}$, we have $\psi(\mathbf{x}) \geq 0$.

Theorem

The Sprague-Grundy function of the turning turltes $(P_{odd}; \to_{\mathrm{D}})$ is given by

$$\psi(\mathbf{x}), \quad \mathbf{x} \in P_{odd}.$$

Example

For $\mathbf{x} = \{2,3,5\} \in P_{odd}$*, the Sprague-Grundy value* $\psi(\mathbf{x})$ *is*

$$\psi(\mathbf{x}) = 2 \oplus 3 \oplus 5 = 4 \neq 0.$$

Hence, the position $\mathbf{x}$ *has a winning strategy. The (unique) winning move is:*

$$\{2,3,5\} \rightarrow_{\mathrm{D}} \{2,3,1\}.$$

The definition 16 means the position set P_{odd} of turning turtles is a "patchwork" of position sets P_n of Sato-Welter games. This is one of our motivation of this study.

Definition

Fix an integer $N \geq 2$. Put

$$P_{N,1} := P_N \cup P_1.$$

Let $n, m \in \{1, N\}$. For $\mathbf{x} \in P_n$ and $\mathbf{y} \in P_m$, we denote $\mathbf{x} \to \mathbf{y}$ if

- $n = m \text{ and } \mathbf{x} {\to_{\mathrm{A}}} \mathbf{y}$; *or*
- $n = N, m = 1 \text{ and } \mathbf{x} \supset \mathbf{y}$.

We call the game $(P_{N,1}; \to)$ the inset game.

Remark

If $N = 2, 3$, then this game is not new (see subsection 1, 2). For $N \geq 4$, this game is new.

Define an injection $f : P_{2,1} \to P_2$ by

$$f(\{x_1, x_2\}) = \{x_1 + 1, x_2 + 1\}, \qquad f(\{x_1\}) = \{0, x_1 + 1\}.$$

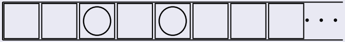
It is straightforward to see the case $N = 2$:

Propsition

The map f is a game isomorphism from $P_{2,1}$ to the image $f(P_{2,1})$. In particular, for $\mathbf{x} \in P_{2,1}$, we have $\varphi_{2,1}(\mathbf{x}) = \varphi_2(f(\mathbf{x}))$.

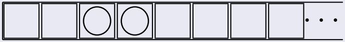
Example

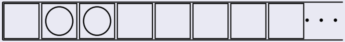
For a position

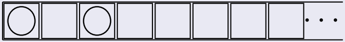
$\mathbf{x} = \{2, 4\} =$  $\in P_{2,1}$

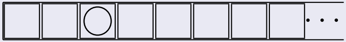
of our game with $N = 2$,

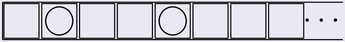
the elements of the option set $\alpha(\mathbf{x})$ *are:*

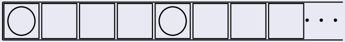
$\{2, 3\} =$ 

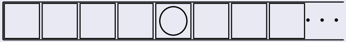
$\{2, 1\} =$ 

$\{2, 0\} =$ 

$\{2\} =$ 

$\{1, 4\} =$ 

$\{0, 4\} =$ 

$\{4\} =$ 

These positions correspond to the positions in Example 12.

Define an injection $f : P_{3,1} \to P_{\text{odd}}$ by

$$f(\{x_1, x_2, x_3\}) = \{x_1, x_2, x_3\}, \qquad f(\{x_1\}) = \{x_1\}.$$

It is straightforward to see the case $N = 3$:

Propsition

The map f is a game isomorphism from $P_{3,1}$ to the image $f(P_{3,1})$. In particular, for $\mathbf{x} \in P_{3,1}$, we have $\varphi_{3,1}(\mathbf{x}) = \psi(f(\mathbf{x}))$.

Example

For a position

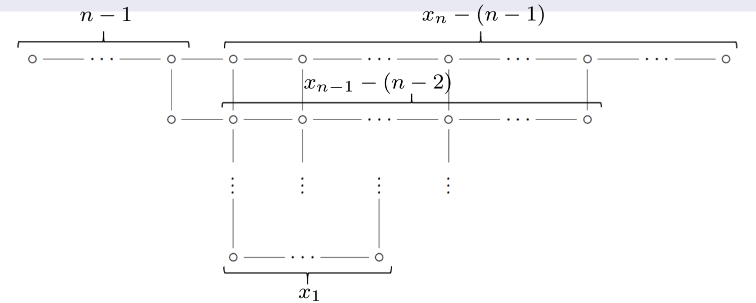
$$\mathbf{x} = \{1,2,4,5\} = \square\bigcirc\bigcirc\square\bigcirc\bigcirc\square\square\cdots \in P_{4,1}$$

of our game with $N = 4$, the elements of the option set $\alpha(\mathbf{x})$ are:

$$\{1,2,4,3\} = \square\bigcirc\bigcirc\bigcirc\bigcirc\square\square\square\cdots$$
$$\{1,2,4,0\} = \bigcirc\bigcirc\bigcirc\square\bigcirc\square\square\square\cdots$$
$$\{1,2,3,5\} = \square\bigcirc\bigcirc\bigcirc\square\bigcirc\square\square\cdots$$
$$\{1,2,0,5\} = \bigcirc\bigcirc\bigcirc\square\square\bigcirc\square\square\cdots$$
$$\{1,0,4,5\} = \bigcirc\bigcirc\square\square\bigcirc\bigcirc\square\square\cdots$$
$$\{0,2,4,5\} = \bigcirc\square\bigcirc\square\bigcirc\bigcirc\square\square\cdots$$
$$\{5\} = \square\square\square\square\square\bigcirc\square\square\cdots$$
$$\{4\} = \square\square\square\square\bigcirc\square\square\square\cdots$$
$$\{2\} = \square\square\bigcirc\square\square\square\square\square\cdots$$
$$\{1\} = \square\bigcirc\square\square\square\square\square\square\cdots$$

Remark

With a position $\mathbf{x} = \{x_1, x_2, \cdots, x_n\} \in P_{N,1}$, *the inset*



can be associated, if $n = N$ and $x_1 < x_2 < \cdots < x_N$. *If, on the other hand,* $n = 1$, *then we attach the* x_1*-chain.*

For $\mathbf{x} = \{x_1, x_2, \cdots, x_n\} \in P_{N,1}$, we put

$$\varphi_{N,1}(\mathbf{x}) := \begin{cases} \sum_i^{\oplus} \left(-1 \,\middle|\, \varphi_{N-1}(\mathbf{x}^{(i)})\right) \oplus \varphi_N(\mathbf{x}) & n = N \\ x_1 & n = 1 \end{cases}$$

where $\mathbf{x}^{(i)} = \mathbf{x} \setminus \{x_i\}$. For $\mathbf{x} \in P_{N,1}$, we have

$$\varphi_{N,1}(\mathbf{x}) \geq 0. \tag{4}$$

Now we can state the main result:

Theorem

The Sprague-Grundy function of the inset game $(P_{N,1}; \to)$ is given by

$$\varphi_{N,1}(\mathbf{x}), \quad \mathbf{x} \in P_{N,1}.$$

Example

For $\mathbf{x} = \{1, 2, 4, 5\} \in P_{4,1}$, *the Sprague-Grundy value* $\varphi_{4,1}(\mathbf{x})$ *is*

$$\begin{aligned}\varphi_{4,1}(\mathbf{x}) &= (-1 \,|\, \varphi_3(2,4,5)) \oplus (-1 \,|\, \varphi_3(1,4,5)) \\ &\quad \oplus (-1 \,|\, \varphi_3(1,2,5)) \oplus (-1 \,|\, \varphi_3(1,2,4)) \oplus \varphi_4(1,2,4,5) \\ &= (-1 \,|\, 0) \oplus (-1 \,|\, 7) \oplus (-1 \,|\, 1) \oplus (-1 \,|\, 4) \oplus 6 = 10.\end{aligned}$$

Hence, the position $\mathbf{x}$ *has a winning strategy. The (unique) winning move is:*

$$\{1, 2, 4, 5\} \to \{0, 1, 4, 5\}.$$

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