Orbit harmonics and cyclic sieving phenomena

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FPSAC 2021
Jan 18, 2022

Cyclic Sieving Phenomena

Definition (Reiner–Stanton–White 04)

Let $C_n = \langle c \rangle$ be a cyclic group acting on a finite set X and let $X(q) \in \mathbb{Z}_{\geq 0}[q]$ be a polynomial. The triple $(X, C_n, X(q))$ exhibits <u>cyclic seiving phenomenon (CSP)</u> if for $\omega = exp(2\pi i/n)$ and $k \geq 0$,

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Definition (Barcelo–Reiner–Stanton 08)

Let $C_{n_1} \times \cdots \times C_{n_m} = \langle c_1 \rangle \times \cdots \times \langle c_m \rangle$ be a product of m cyclic groups acting on a finite set X and let $X(q_1, \ldots, q_m) \in \mathbb{Z}_{\geq 0}[q_1, \ldots, q_m]$ be a polynomial in $q_1, \ldots, q_m$. The triple $(X, C_{n_1} \times \cdots \times C_{n_m}, X(q_1, \ldots, q_m))$ exhibits m-ary cyclic seiving phenomenon (m-ary CSP) if for $\omega_1 = exp(2\pi i/n_2)$,$\ldots \omega_m = exp(2\pi i/n_m)$ and $k_1, \ldots, k_m, \geq 0$,

$$|X^{(c_1^{k_1}, \ldots, c_m^{k_m})}| = X(\omega_1^{k_1}, \ldots, \omega_m^{k_m}).$$

Cyclic Sieving Phenomena (example)

Theorem (Reiner–Stanton–White 04)

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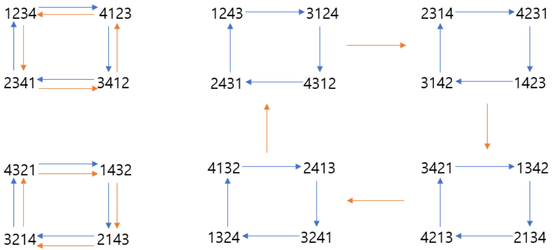
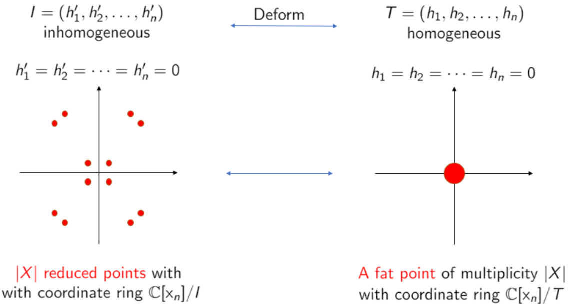
Theorem (Barcelo–Reiner–Stanton 08)

The triple $(\mathfrak{S}_n, C_n \times C_n, X_n(q,t))$ *exhibits biCSP, where* $X_n(q,t) = \sum_{\lambda \vdash n} f^\lambda(q) f^\lambda(t)$.



Orbit harmonics



X: a finite set in $\mathbb{C}^n$ which is closed under $\mathfrak{S}_n \times C$, where

- symmetric group $\mathfrak{S}_n$ acts on $\mathbb{C}^n$ by coordinate permutation
- a cyclic group C acts on $\mathbb{C}^n$ by scaling a root of unity

Then we have an isomorphism as $\mathfrak{S}_n \times C$-modules,

$$\mathbb{C}[X] \cong \mathbb{C}[x_n]/I \cong \mathbb{C}[x_n]/T.$$

CSP generating theorem

Sieving generating theorem (O.–Rhoades 21)

Let $X \subseteq \mathbb{C}^n$ be a finite set with $\mathfrak{S}_n \times C_k$ acting on it. Let

$$\mathrm{grFrob}(\mathbb{C}[x_n]/T(X); q) = \sum_{\lambda} c_\lambda(q) s_\lambda(x).$$

Then we have the followings.

1. $(X, C \times C_k, X(q,t))$ exhibits biCSP where $C = \langle (12 \cdots n) \rangle \subseteq \mathfrak{S}_n$, and
$$X(q,t) = \sum_{\lambda} c_\lambda(t) f^\lambda(q).$$
2. For a subgroup $G \subseteq \mathfrak{S}_n$, $(X/G, C_k, X(q))$ exhibits CSP where
$$X(q) = \mathrm{Hilb}((\mathbb{C}[x_n]/T(X))^G; q) = \sum_{\lambda} c_\lambda(q) \langle S^\lambda \downarrow_G, 1_G \rangle.$$

Examples

Let $\mathcal{A}_{n,k} := \{(a_1, \ldots, a_n) : a_i \in \{\omega, \omega^2, \ldots, \omega^k\}\}$ be a functional locus, where $\omega = \exp(2\pi i/k)$. Then we have the following sieving results.

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- A surjective functional locus + $G = H_{n/2}$ gives a CSP for graphs with no isolated vertices. (O.–Rhoades 21).

The module of Garsia–Haiman

To each partition μ of n, one associates a bigraded $\mathfrak{S}_n$-module called the Garsia–Haiman module as follows. For each injective tableau T, we assign a point $p_T \in \mathbb{C}^{2n}$ by letting i^{th} and $(n+i)^{th}$ coordinates of p_T record the position of i in T:

$$p_T = (\omega^{x_T(1)}, \ldots, \omega^{x_T(n)}, \zeta^{y_T(1)}, \ldots, \zeta^{y_T(n)}),$$

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For example,

$$\begin{array}{|c|c|c} \hline 4 & 1 & \\ \hline 2 & 3 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \longleftrightarrow (\omega^2, \omega^1, \omega^2, \omega^1, \omega^3, \zeta^2, \zeta^1, \zeta^1, \zeta^2, \zeta^1)$$

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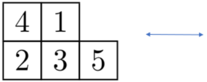


$$(\omega^2, \omega^1, \omega^2, \omega^1, \omega^3, \zeta^2, \zeta^1, \zeta^1, \zeta^2, \zeta^1)$$

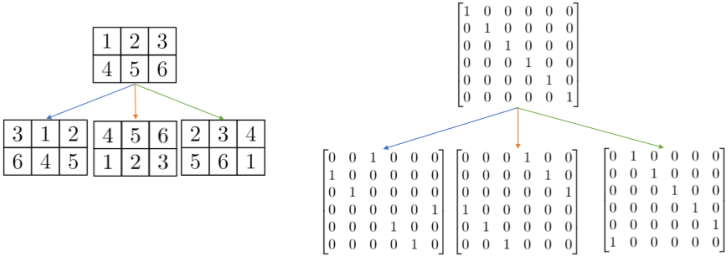
The point locus $X_\mu := \{p_T : T \in \mathrm{Inj}(\mu)\}$ possesses a natural $\mathfrak{S}_n$ action that acts diagonally on X_μ.

Theorem (Garsia–Haiman 96, Haiman 01)

The graded Frobenius image of $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/\mathbf{T}(X_\mu)$ is the Macdonald polynomial $\widetilde{H}_\mu(\mathbf{x}; q, t)$.

A triCSP for $\mathfrak{S}_n$

For $\mu = (a^b)$, the point locus $X_\mu \subseteq \mathbb{C}^{2n}$ has $\mathfrak{S}_n \times C_a \times C_b$ action.



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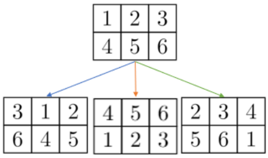

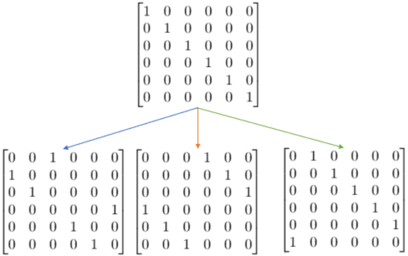

Theorem (O. 22)

The triple $(X_n, C_a \times C_b \times C_n, X_{(a^b)}(q,t,z))$ exhibits triCSP, where

$$X_{(a^b)}(q,t,z) = \sum_{\lambda \vdash ab} \tilde{K}_{\lambda,(a^b)}(q,t) f^\lambda(z).$$

Thank you!