Naruse Hook length formula for linear extensions of mobile posets

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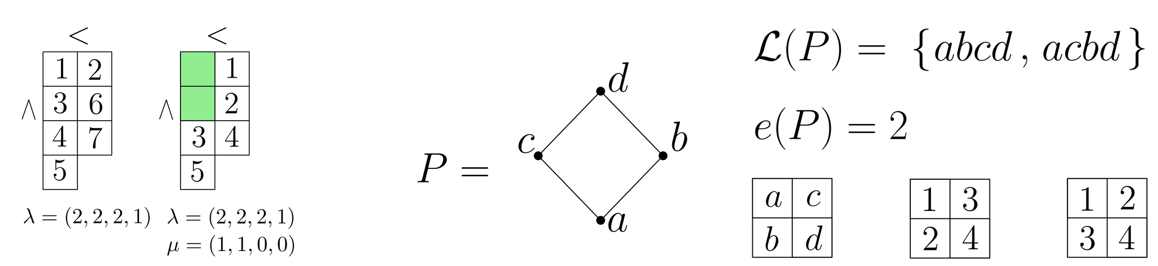
Objective

- Extend the Naruse Hook Length Formula to *mobile posets*
- Find a q-analogue of the NHLF for mobile posets

Standard Young Tableaux

- A standard Young tableau (SYT) is a filling of λ with $1, \dots, n$ such that it is increasing in rows and columns.
- A skew shape is a pair of partitions (λ, μ) such that $\mu \subseteq \lambda$, denoted as λ/μ. A *skew SYT* is a filling of λ/μ with integers $1 \dots n$ increasing in rows and columns.



Linear Extensions

- A linear extension of a poset P is a linear order of elements compatible with the order P. Let $\mathcal{L}(P)$ the set of all linear extensions of P and $e(P) = |\mathcal{L}(P)|$.
- $e(P(\lambda/\mu)) = |\mathrm{SYT}(\lambda/\mu)|$

Hook Length and Naruse's Hook Length Formula

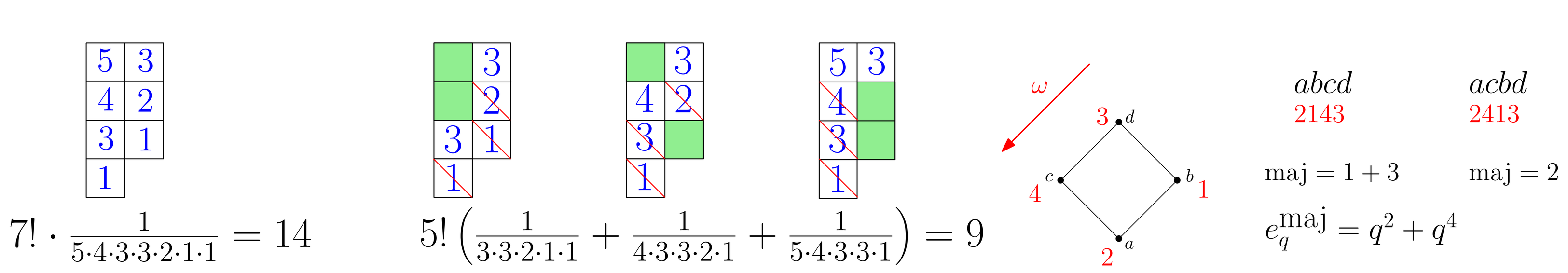
Theorem 1. (Frame-Robinson-Thrall, 1954) For partition λ of n, we have

$$|\mathrm{SYT}(\lambda)| = n! \prod_{(i,j)\in[\lambda]} \frac{1}{h(i,j)}$$

where $h(i,j) = \lambda_i - i + \lambda'_j - j + 1$ is the hook length of the square (i,j)

Theorem 2. (Naruse, 2014) For a skew shape λ/μ, we have

$$|\mathrm{SYT}(\lambda/\mu)| = |\lambda/\mu|! \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{u\in[\lambda]\setminus D} \frac{1}{h(u)}. \qquad \text{(NHLF)}$$



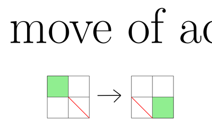
Theorem 3. (Morales, Pak, Panova 15)

$$\frac{e_q^{\mathrm{maj}}(P_{\lambda/\mu}, \omega)}{\prod_{i=1}^n (1-q^i)} = s_{\lambda/\mu}(1, q, q^2, \dots) = \sum_{S\in\mathcal{E}(\lambda/\mu)} q^{w(D)} \prod_{(i,j)\in[\lambda]\setminus S} \frac{1}{1-q^{h(i,j)}}$$

where $w(D) = \sum_{u\in Br(D)} h(u)$ is the sum of hook-lengths of broken diagonals.

Excited Diagram (Ikeda-Naruse)

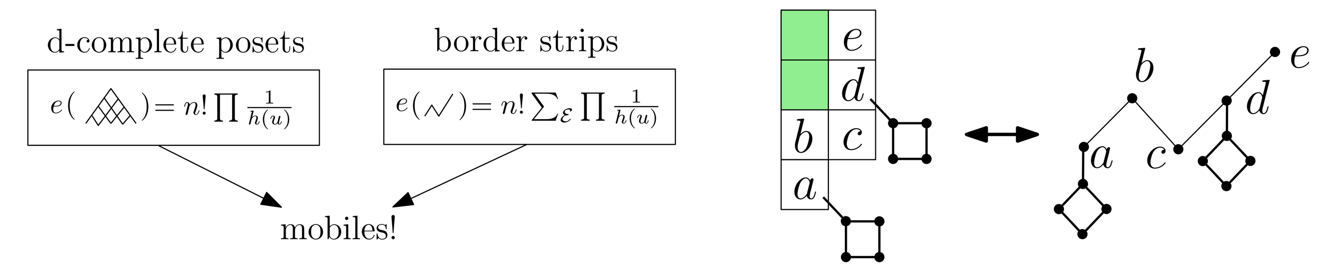
An excited move in λ/μ is a sliding move of active cell from (i,j) to $(i+1, j+1)$.



An excited diagram of λ/μ is a subdiagram of $[\lambda]$ obtained from the Young diagram of μ after a sequence of excited moves.

Mobile Posets

Question: Are there other posets with Naruse hook length formula, generalizing formula for $e(P)$?



- d-complete posets are a large class of posets that includes (shifted) Young diagrams and rooted tree posets with HLF (Proctor)
- A mobile poset P is a poset obtained from a border strip λ/μ by hanging from it $d-$complete posets. (Garver-Grosser-Matherne-Morales)
- GGMM found a formula for $e(P)$ for mobile posets as a determinant of products of hook lengths (analogue of Jacobi–Trudi formula for border strips)

NHLF for Mobiles

- For (i,j) in λ/μ let $h'(i,j) = h(i,j) + \sum_{a\geq i, b\geq j} p_{a,b}$ where $p_{a,b}$ is the size of the d-complete poset hanging on $(a,b) \in P_{\lambda/\mu}$

Theorem 4. (P, 2021+) Let $(P_{\lambda/\mu}(\mathbf{p}), \omega)$ be a free-standing mobile poset of size n and ω is the reversed Schur labeling. Then

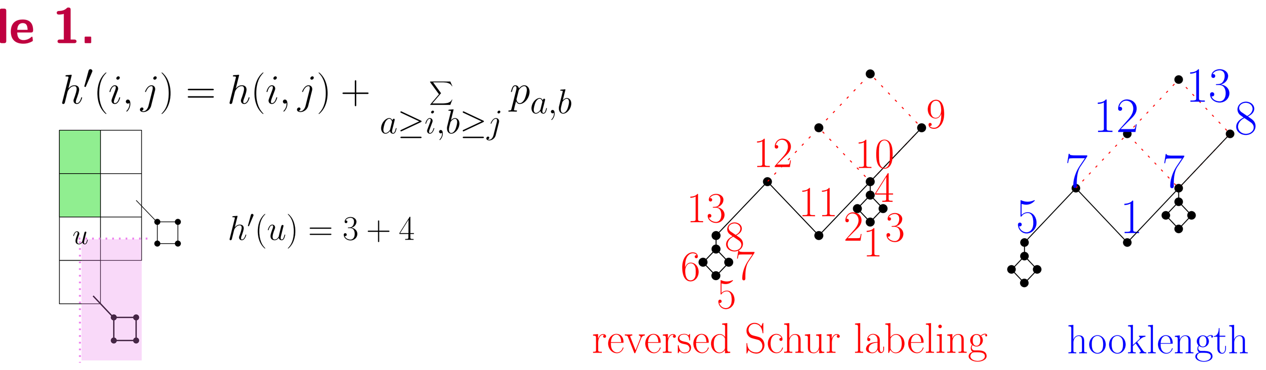
$$\frac{e_q^{\mathrm{maj}}(P_{\lambda/\mu}, \omega)}{\prod_{i=1}^n (1-q^i)} = \prod_{v\in\mathbf{p}} \frac{1}{1-q^{h(v)}} \sum_{D\in\mathcal{E}(\lambda/\mu)} q^{w'(D)} \prod_{u\in[\lambda]\setminus D} \frac{1}{1-q^{h'(u)}}$$

and $w'(D) = \sum_{u\in\mathrm{Br}(D)} h'(u)$ is the sum of hook-lengths of broken diagonals.

Corollary 1. (P, 2021+)

$$e(P_{\lambda/\mu}(\mathbf{p})) = \frac{n!}{H(\mathbf{p})} \sum_{D\in(\lambda/\mu)} \prod_{(i,j)\in\gamma} \frac{1}{h'(i,j)}.$$

Example 1.



$$e_q^{\mathrm{maj}}(P) = \frac{[13]!}{[1]^2[2]^4[3]^2}\left(\frac{q^{12}}{[1][5][6][7]^2} + \frac{q^{18}}{[5][6][7]^2[12]} + \frac{q^{24}}{[5][7]^2[12][13]}\right)$$
$$= q^{61} + 2q^{60} + 6q^{59} + 11q^{58} + \cdots + 6q^{14} + 2q^{13} + q^{12}.$$

$$e(P) = \frac{13!}{1^2\cdot2^4\cdot3^2}\left(\frac{1}{5\cdot6\cdot7^2} + \frac{1}{5\cdot6\cdot7^2\cdot12} + \frac{1}{5\cdot7^2\cdot12\cdot13}\right) = 33000,$$

where $[m] = 1 - q^m$.

Bounds for number of linear extensions

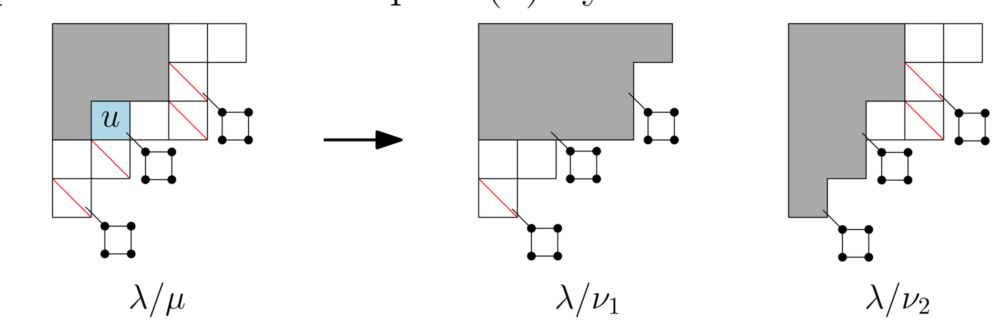
Corollary 2. For any mobile poset $e(P_{\lambda/\mu}(\mathbf{p}))$ of size n,

$$\frac{n!}{H(\mathbf{p})\prod_{u\in[\lambda/\mu]} h'(u)} \leq e(P_{\lambda/\mu}(\mathbf{p})) \leq |\mathcal{E}(\lambda/\mu)| \cdot \frac{n!}{H(\mathbf{p})\prod_{u\in[\lambda/\mu]} h'(u)}$$

where $[\lambda/\mu]$ is the border strip of the mobile poset.

Method of Proof

We follow the proof for border strip of (2) by Morales-Pak-Panova (2019)



Lemma 1. For a labeled mobile poset $(P_{\lambda/\mu}(\mathbf{p}), \omega)$, where ω is a reverse Schur labeling,

$$e_q^{\mathrm{maj}}(P_{\lambda/\mu}, \omega) = \sum_{\mu\to\nu} q^{|P_{\lambda/\nu_1}|} e_q^{\mathrm{maj}}(P_{\lambda/\nu}, \omega_\nu)$$

- This lemma is proved using a generalization of Stanley's theory of P-partitions.

Lemma 2.

$$(1-q^n)\cdot H_{\lambda/\mu}(q) = \sum_{\mu\to\nu} q^{|P_{\lambda/\mu_1}|} \cdot H_{\lambda/\nu^1}(q) \cdot H_{\lambda/\nu^2}(q)$$

where $H_{\lambda/\mu}(q)$ is the sum on the RHS of Theorem 4.

- Proved by evaluating $x_i = \lambda_i - i + 1 - \sum_{a<i} p_{a,b}$, and $y_j = j - \lambda'_j - \sum_{b\geq j} p_{a,b}$ in the Pieri-Chevalley formula

$$F_{\lambda/\mu}(\mathbf{x}|\mathbf{y}) = \frac{1}{x_1 - y_1} \sum_{\mu\to\nu} F_{\lambda/\nu}(\mathbf{x}|\mathbf{y})$$

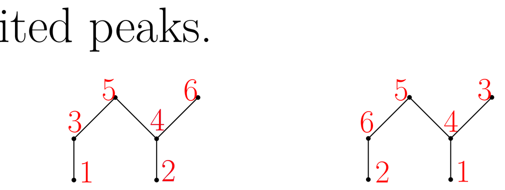
where $F_{\lambda/\mu}(\mathbf{x}|\mathbf{y}) := \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in[\lambda]\setminus D} \frac{1}{x_i - y_j}$.

Skew d-complete Poset (RPP)

Theorem 5. (Naruse-Okada, 2019) Let P be a d-complete poset and F an order filter of P. Then the multivariate generating function of $(P \setminus F)$-partition is

$$e_q^{\mathrm{maj}}(P\setminus I) = \prod_{i=1}^n (1-q^i) \sum_{D\in\mathcal{E}(P\setminus I)} \frac{\prod_{v\in\mathrm{pk}(D)} q^{h(v)}}{\prod_{v\in P\setminus D}(1-q^{h(v)})}$$

where $\mathrm{pk}(D)$ is a set of excited peaks.



Other results

- Extension of Stanley's P-partition theory for fixing positions of linear extensions.
- q-analogue for inversion index statistic for *mobile tree posets.*

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- GaYee Park, *Naruse hook formula for linear extensions of mobile posets (in preparation).*