Gröbner geometry for skew-symmetric matrix Schubert varieties

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Borel group matrix orbits

- $\mathsf{M}_n, \mathsf{M}_n^{\mathsf{ss}}, \mathsf{M}_n^{\mathsf{s}}$ the spaces of all matrices/skew-symmetric matrices/symmetric matrices over an alg. closed field $\mathbb{K}$
- B_n the group of $n \times n$ upper triangular matrices over $\mathbb{K}$
- $B_n \times B_n$ acts on M_n: $(b_1, b_2) \cdot A = b_1 A b_2^t$
- Restricts to a B_n-action on $\mathsf{M}_n^{\mathsf{ss}}$ and $\mathsf{M}_n^{\mathsf{s}}$: $b \cdot A = bAb^t$.

What are the orbits of these actions? Focus on invertible matrices; can reduce to this case for most things we care about.

- $B_n \times B_n$-orbits on GL_n = orbits of permutation matrices
- B_n-orbits on $\mathsf{GL}_n \cap \mathsf{M}_n^{\mathsf{s}} \longleftrightarrow$ involutions on $[n] := \{1, 2, \ldots, n\}$
- B_n-orbits on $\mathsf{GL}_n \cap \mathsf{M}_n^{\mathsf{ss}} \longleftrightarrow$ fixed-point-free (fpf) involutions on $[n]$

Matrix Schubert varieties

The closures of these orbits are affine varieties: matrix Schubert varieties (ordinary/symmetric/skew-symmetric).

- $X_w :=$ the orbit closure of permutation matrix w (or X^{s}_{y}, X^{ss}_{z} in symmetric/skew-symmetric case).
- A matrix Schubert variety is defined by upper-left rank conditions, e.g.

$$X_{\begin{bmatrix} 0&1&0&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&0&1&0 \end{bmatrix}} = \{A \in \mathrm{M}_4 : A_{11} = 0, \operatorname{rank} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} \leq 1\}.$$

- Defined as sets by insisting certain minors vanish
- Fulton: the prime ideal $I(X_w)$ is generated by these minors; Knutson-Miller: they also form a Gröbner basis with a suitable term order.

Skew-symmetric matrix Schubert varieties

- Indexed by $\mathcal{I}_n^{\mathrm{FPF}}$, the set of fixed-point-free involutions in S_n, e.g. $\mathcal{I}_4^{\mathrm{FPF}} = \{(1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$.
- Again defined as sets by upper-left rank conditions / vanishing of minors, but now minors need not generate a prime ideal:

$$\operatorname{rank} \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix} \leq 1 \Leftrightarrow \det \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix} = x^2 = 0.$$

- Use pfaffians: $\mathrm{pf}(A) = \sum_{z \in \mathcal{I}_n^{\mathrm{FPF}}} (-1)^{\ell^{\mathrm{FPF}}(z)} \prod_{\text{cycles } (i,j) \text{ of } z} A_{ij}$, $\ell^{\mathrm{FPF}}(z)$ the rank of z in Bruhat order restricted to $\mathcal{I}_n^{\mathrm{FPF}}$.
- $\mathrm{pf}(A)^2 = \det(A)$ if A is skew-symmetric.

Ideals of skew-symmetric matrix Schubert varieties

Let $\mathcal{U}$ be a skew-symmetric matrix of indeterminates u_{ij}, $i > j$.
Write A_{RC} for the submatrix of A in rows R, columns C.

Theorem (Marberg-Pawlowski)

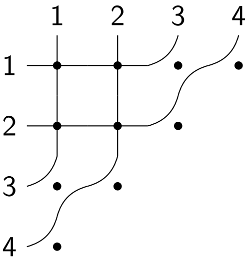
The prime ideal of the B-orbit of $A \in \mathrm{M}_n^{ss}$ is generated by $\mathrm{pf}(\mathcal{U}_{RR})$ for all nonempty $R \subseteq [n]$ such that there are $i, j \in [n]$ with $R \subseteq [i]$, $|R \cap [j]| > \mathrm{rank}\, A_{[i][j]}$.

Initial ideals

- $\text{init}(f) =$ leading term of f in reverse lex order
- $\text{init}(I) =$ ideal generated by $\{\text{init}(f) : f \in I\}$
- If S generates I, not always true that $\{\text{init}(f) : f \in S\}$ generates $\text{init}(I)$; if it does, S is a Gröbner basis.
- Can calculate polynomials representing cohomology or K-theory classes of X_z^{ss} just from $\text{init}\, I(X_z^{ss})$.

Fpf involution pipe dreams

An fpf involution pipe dream for $z = (1,3)(2,4) = 3412 \in \mathcal{I}_4^{\mathrm{FPF}}$:



An fpf involution pipe dream for an fpf involution $z \in S_n$ is...

- a symmetric tiling of $[n] \times [n]$ with tiles  and 
- where the i^{th} pipe on the left is the $z(i)^{\text{th}}$ pipe on the top...
- and no pair of pipes crosses more than once.

Identify an fpf involution pipe dream with its set of $+$ tiles strictly below the diagonal, e.g. $\{(2,1)\}$ above.

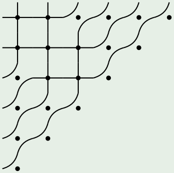
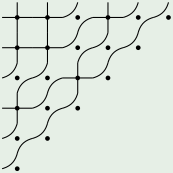
Primary decompositions

Theorem (Marberg–Pawlowski)

For $z \in \mathcal{I}_n^{\mathrm{FPF}}$, init $I(X_z^{ss}) = \bigcap_D (u_{ij} : (i,j) \in D)$ *where* D *runs over the set of fpf involution pipe dreams of* z*.*

Example

$z = (1,3)(2,5)(4,6)$ has two fpf involution pipe dreams:

$\{(2,1),(3,2)\} =$  $\{(2,1),(4,1)\} =$ 

So init $I(X^{ss}_{(1,3)(2,5)(4,6)}) = (u_{21}, u_{32}) \cap (u_{21}, u_{41}) = (u_{21}, u_{32}u_{41})$.

Proofs and corollaries

- Proof uses an explicit Gröbner basis of $I(X_z^{ss})$ (not the generating set of Pfaffians presented earlier!)
- Inductive proof, but different from Knutson–Miller; obtain a new proof of their results on matrix Schubert varieties.

Corollaries à la Knutson–Miller:

- $I(X_z^{ss})$ is the Stanley-Reisner ideal of a shellable simplicial complex
- Torus-equivariant cohomology class of X_z^{ss} is $\sum_D \prod_{(i,j)\in D}(x_i + x_j)$ where D runs over fpf involution pipe dreams of z (something like a Schubert polynomial)
- Similar formula for equivariant K-theory class in terms of nonreduced fpf involution pipe dreams

Hope: do all this for symmetric matrix Schubert varieties (connects to L-R rule for K-theory of Lagrangian Grassmannian).