Proof of the Cameron and Fon-Der-Flaass periodicity conjecture

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Rowmotion

- Let P be a finite poset and $J(P)$ its set of order ideals.
- **Rowmotion** is the permutation $\psi\colon J(P) \to J(P)$ sending an order ideal I to the smallest order ideal $\psi(I)$ containing the minimal elements of $P \setminus I$.

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- This morning: $P = \mathbf{a} \times \mathbf{b} \times \mathbf{c}$, a product of three chain posets.

Motivation

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- Early work by Brouwer+Schrijver (1974) and Cameron+Fon-Der-Flaass (1995) is mostly unmotivated.
- Finally attracted attention *circa* 2010 after work of Panyushev (2009) in the context of root systems.
- Now a very active area, with mysterious connections to cluster algebras (Shen+Weng 2020), quantum Schubert calculus (Buch+Wang 2021), quiver representations (Garver+Patrias+Thomas 2018), Auslander algebras (Marczinzik+Thomas+Yıldırım 2022), and K-theoretic Schubert calculus.

$c = 1$

Theorem (Brouwer+Schrijver 1974)

The order of ψ *on* $J(\mathbf{a} \times \mathbf{b} \times \mathbf{1})$ *is* $a + b$.
More precisely, for $I \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{1})$, $|\mathcal{O}(I)|$ *divides* $a + b$ *and* $|\mathcal{O}(\emptyset)| = a + b$.

Theorem (Striker+Williams 2012)

$(J(\mathbf{a} \times \mathbf{b} \times \mathbf{1}), \psi, f(q))$ *exhibits cyclic sieving, where* $f(q)$ *is the* q*-enumerator for order ideals by cardinality.*

Theorem (Cameron+Fon-Der-Flaass 1995)

The order of ψ *on* $J(\mathbf{a} \times \mathbf{b} \times \mathbf{2})$ *is* $a + b + 1$.
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$c = 3$

Conjecture (Dilks+P+Striker 2017)

The order of ψ on $J(\mathbf{a} \times \mathbf{b} \times \mathbf{3})$ is $a + b + 2$.

No obvious CSP

$c \geq 4$

Order of ψ generally greater than $a + b + c - 1$ but unknown.

No good bounds on order known. (For $a = b = c = 4$, order is $11 \cdot 3$; for $a = 4$, $b = c = 11$, order is $\geq 309 \cdot 25$.)

No obvious CSP

Cameron+Fon-Der-Flaass Conjecture

Conjecture (Cameron+Fon-der-Flaass 1995)

If $a + b + c - 1$ *is prime, then* $a + b + c - 1$ *divides every* ψ*-orbit size on* $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$.

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Theorem (Patrias+P 2020)

The conjecture is true. More generally, with no primality condition, we have

$$\gcd(a + b + c - 1, |\mathcal{O}(I)|) > 1$$

Let $\lambda \subseteq \nu$ be partitions. An **increasing tableau** of shape $\nu \setminus \lambda$ is a filling of the skew Young diagram $\nu \setminus \lambda$ by positive integers with strictly increasing rows and columns.

$$\lambda = (3,2,1), \nu = (4,4,3), T = \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 2 & 3 \\ \hline & 2 & 4 \\ \cline{1-3} \end{array}$$

K-theoretic jeu de taquin (Thomas+Yong 2009) rectifies this to an increasing tableau of partition shape (computing K-theoretic Schubert structure coefficients).

K-promotion of increasing tableaux

$\mathrm{Inc}^q(\lambda) = \{$increasing tableaux of shape λ with entries in [q]$\}$

$$\mathrm{Inc}^5(2 \times 3) \ni T = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 4 & 5 \\ \hline \end{array}$$

K-promotion recipe (P 2014):

1. Delete 1

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Equivariant bijection

Theorem (Dilks+P+Striker 2017)

There is an equivariant bijection between

$$(J(\mathbf{a} \times \mathbf{b} \times \mathbf{c}), \psi)$$

and

$$(\mathrm{Inc}^{a+b+c-1}(a \times b), \psi)$$

Frames of increasing tableaux

The **frame** of $T \in \mathrm{Inc}^q(a \times b)$ is the union of the boxes in the first/last row and the first/last column.

Example

If $T = $

1	2	4	7
3	5	6	8
5	7	8	10
7	9	10	11

, then $\psi^{11}(T) = $

1	2	4	7
3	4	6	8
5	6	8	10
7	9	10	11

.

The least k such that $\psi^k(T) = T$ is $k = 33$.

Theorem (P 2017)

For $T \in \mathrm{Inc}^q(m \times n)$, *we have* $\mathrm{Frame}(\psi^q(T)) = \mathrm{Frame}(T)$.

Theorem (Patrias+P 2020)

Suppose the ψ-orbit of $T \in \mathrm{Inc}^q(a \times b)$ has cardinality k. Then k shares a prime divisor with q. (Unless $q = a + b - 1$, in which case $k = 1$.)

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Proof.

Suppose $\gcd(k, q) = 1$. Then by the frame theorem, $\mathrm{Frame}(T) = \mathrm{Frame}(\psi(T))$. By analysis of the promotion operator, every frame box of such a tableau must participate in a swap, so the frame entries increase consecutively from upper-left to lower-right. So T is the unique element of $\mathrm{Inc}^{a+b-1}(a \times b)$. □

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Corollary (Patrias+P 2020, Conj: Cameron+Fon-der-Flaass 1995)

If $p = a + b + c - 1$ is prime, then the length of every ψ-orbit on $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ is a multiple of p.

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For standard tableaux, promotion orbits carry information about the geometry of Grassmannians. What is the more general geometry for plane partitions?

Thank you!!