A Refinement of the Murnaghan-Nakayama Rule by Descents for Border Strip Tableaux

Background

$$\chi^\lambda(\rho) = \sum_{B\in \mathrm{BST}(\lambda,\rho)} (-1)^{\mathrm{ht}(B)}$$

$\mathfrak{S}_n$ - character

(Murnaghan - Nakayama)

$$\chi^\lambda(k^{n/k}) = \epsilon_{\lambda,k} \cdot |\mathrm{BST}(\lambda,k)|$$

rectangular partition

± 1

(James - Kerber)

$$\chi^\lambda(k^{n/k}) = f^\lambda(\xi,1)$$

(Springer)

Definitions

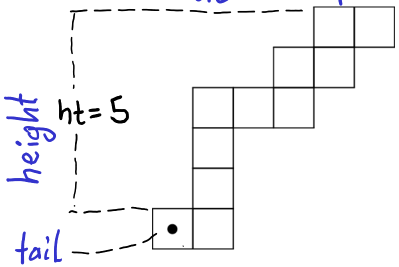
Refined fakedegree polynomial

$$f^\lambda(q,t) := \sum_{T\in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} t^{|\mathrm{DES}(T)|}$$

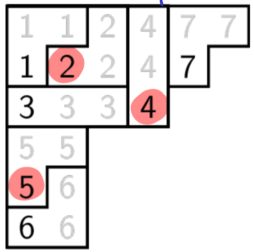
Standard Young tableaux of shape λ

sum of descents

A borderstrip



A borderstrip tableau in BST(λ,3)

$B =$ 

Descent: i+1 is in a lower row than i

DES(B) = {2,4,5}

Theorem

Provided that BST(λ,k) ≠ ∅ we have:

$$f^\lambda(\xi,t) = \epsilon_{\lambda,k} \cdot \sum_{B\in \mathrm{BST}(\lambda,k)} t^{k\cdot|\mathrm{DES}(B)|+\mathrm{ht}(B^1)}$$

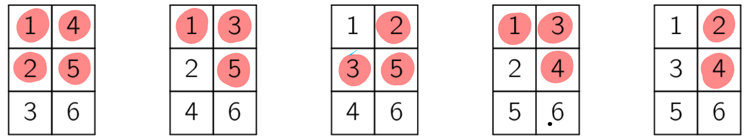
primitive k-th root of unity

± 1

borderstrip tableaux with stripsize k and shape λ

strip containing 1

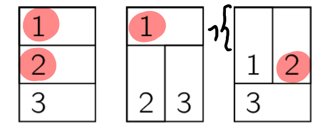
Example



$$f^{222}(q,t) = q^{12}t^4 + q^9t^3 + q^{10}t^3 + q^8t^3 + q^6t^2$$

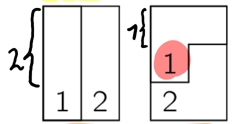
prim. root	$f^{222}(\cdot,t)$
$1^{st}=1$	$t^4+3t^3+t^2$
$2^{nd}=-1$	$t^4+t^3+t^2$
3^{rd}	t^4+t^2
6^{th}	$\underbrace{t^4-2t^3+t^2}_{\mathrm{BST}(222,6)=\emptyset}$

BST(λ,2) :



$t^{2\cdot 2+0}$ $t^{2\cdot 1+0}$ $t^{2\cdot 1+1}$

BST(λ,3) :



$t^{3\cdot 0+2}$ $t^{3\cdot 1+1}$

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