Fundamental expansion of the quasisymmetric Macdonald polynomials

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Overview:

- symmetric and quasisymmetric functions
- Macdonald polynomials
- permuted basement Macdonald polynomials and quasisymmetric Macdonald polynomials
- Fundamental expansion
- tableaux formula
- sketch of proofs

quasisymmetric functions

- a composition $\alpha = \alpha_1 \cdots \alpha_k \models n$ is a vector $\mathbb{Z}^k_{>0}$ whose sum is n
- each $\alpha \models n$ corresponds to a subset $S = \{i_1, \ldots, i_{k-1}\} \subseteq [n-1]$:

$$\alpha = 141 \models 6 \qquad \Rightarrow \qquad S = \{1,5\} \subseteq [5]$$

- define the algebra of quasisymmetric functions:

$$\mathrm{QSym} = \mathrm{QSym}^{\mathbf{0}} \oplus \mathrm{QSym}^{\mathbf{1}} \oplus \cdots ,$$

where $\mathrm{QSym}^{\mathbf{n}} = \mathrm{span}_{\mathbb{Q}}\{M_S | S \subseteq [n-1]\}$ given by the monomial quasisymmetric functions:

$$M_S = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$

- another basis: fundamental quasisymmetric functions F_α:

$$F_S = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n \\ j \in S \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\beta \leq \alpha} M_\beta = \sum_{S \subseteq S'} M_{S'}$$

where $\beta \leq \alpha$ means β is a refinement of α.

Example:

$$\begin{aligned} F_{13} &= M_{13} + M_{121} + M_{112} + M_{1111} \\ &= M_{\{1,4\}} + M_{\{1,3,4\}} + M_{\{1,2,4\}} + M_{\{1,2,3,4\}} \end{aligned}$$

Macdonald polynomials

- Let $\Lambda \cong \Lambda_{\mathbb{Q}}(q,t)$ be the ring of symmetric polynomials with parameters q, t over $\mathbb{Q}$
- The standard inner product on Λ, with parameters q, t, is $\langle \cdot, \cdot \rangle \cong \langle \cdot, \cdot \rangle_{q,t}$, is defined by: $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda \prod_i \dfrac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$
- the symmetric Macdonald polynomials $P_\lambda(X;q,t) \in \Lambda$ are uniquely determined by Macdonald's triangularity and normalization axioms:
  i. upper triangular with respect to $\{m_\lambda\}$:

$$P_\lambda(X;q,t) = m_\lambda(X) + \sum_{\mu<\lambda} c_{\mu\lambda}(q,t) m_\mu(X)$$

  ii. orthogonal basis for Λ: $\langle P_\lambda, P_\mu \rangle = 0$ if $\lambda \neq \mu$

- Example: $P_{(2,1)}(X;q,t) = m_{(2,1)} + \dfrac{(1-t)(2+q+t+2qt)}{1-qt^2} m_{(1,1,1)}$.
- the nonsymmetric Macdonald polynomials $E_\mu(X;q,t)$ were introduced by Cherednik, Macdonald, and Opdam as a tool to study P_λ. They are indexed by weak compositions and form a basis for $\mathbb{Q}[X](q,t)$

quasisymmetric Macdonald polynomials

- permuted basement Macdonald polynomials $E_{\mu}^{\sigma}(x_1, \ldots, x_n; q, t)$ are an extension of the E_{μ}'s introduced by Ferreira, Alexandersson; here $\mu = (\mu_1, \ldots, \mu_n)$ and $\sigma \in S_n$
- For $\lambda = (\lambda_1, \ldots, \lambda_n)$ a partition, define $\mathrm{inc}(\lambda) = (\lambda_n, \ldots, \lambda_1)$. Then

  $$P_{\lambda}(x_1, \ldots, x_n; q, t) = \sum_{\alpha \in S_n \cdot \lambda} E_{\mathrm{inc}(\lambda)}^{\beta(\alpha)}(x_1, \ldots, x_n; q, t)$$

  where $\beta(\alpha)$ is the **permutation of longest length** such that $\beta(\alpha) \circ \alpha = \mathrm{inc}(\alpha)$ (Corteel-M-Williams '18)
- this motivated the definition of a new quasisymmetric polynomial, indexed by (strong) compositions γ:

  $$G_{\gamma}(X; q, t) = \sum_{\alpha : \alpha^{+} = \gamma} E_{\mathrm{inc}(\alpha)}^{\beta(\alpha)}(X; q, t)$$

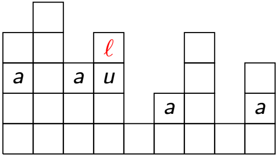
  (Corteel-Haglund-M-Mason-Williams '20)
- In particular, we recover the quasisymmetric Schur polynomial:

  $$G_{\gamma}(X; 0, 0) = \mathrm{QS}_{\gamma}(X),$$

  and by definition, $P_{\lambda}(X; q, t) = \sum_{\gamma \in S_n \cdot \lambda} G_{\gamma}(X; q, t)$

Haglund–Haiman–Loehr tableaux statistics

The diagram of a composition $\nu = (4, 5, 3, 4, 1, 2, 4, 1, 3)$:



- **fillings** of T must be non-attacking:

  no repeated entries in configurations ☐ ··· ☐ or $\overset{\square}{}\ \cdots\ \underset{\square}{}$

- **major index:** $Des(T) = \{u : T(u) > T(\text{South}(u))\}$

$$\text{maj}(T) = \sum_{u \in \text{Des}(T)} \text{leg}(u) + 1.$$

- **triples:** **Type A:** $\begin{array}{|c|}\hline x \\ \hline z \\ \hline \end{array} \quad \overset{\boxed{y}}{}$ and **Type B:** $\underset{\boxed{y}}{} \quad \begin{array}{|c|}\hline x \\ \hline z \\ \hline \end{array}$

  a triple is a coinversion if entries are (cyclically) increasing in clockwise orientation for Type A or in counterclockwise orientation for Type B

- **coinversions:** coinv(T) is the total number of coinversion triples

combinatorial formula for G_γ

- Let $\sigma = \beta(\gamma)$, the permutation of longest length such that $\sigma \circ \gamma = \mathrm{inc}(\gamma)$
- Define $\mathrm{NAT}(\gamma)$ to be the set of non-attacking fillings whose entries in row 1 are **order equivalent to** σ when read L to R.

Example:

$$T = \begin{array}{cccc} & & & 5 \\ & 2 & 7 & 5 \\ & 4 & 1 & 2 \\ 5 & 8 & 4 & 2 \end{array} \in \mathrm{NAT}((4,3,1,3))$$

$$\begin{aligned}
\alpha &= (0,4,0,3,1,0,0,3) \\
\mathrm{inc}(\alpha) &= (0,0,0,0,1,3,3,4) \\
\beta(\alpha) &= (7,6,3,1,5,8,4,2) \\
\gamma &= \alpha^+ = (4,3,1,3) \\
\mathrm{inc}(\gamma) &= (1,3,3,4) \\
\sigma &= \beta(\alpha^+) = (3,4,2,1) \\
x^T &= x_1 x_2^3 x_4^2 x_5^3 x_7 x_8 \\
\mathrm{maj}(T) &= 3+1 = 4 \\
\mathrm{coinv}(T) &= 2
\end{aligned}$$

Then we have:

$$G_\gamma(X;q,t) = \sum_{T \in \mathrm{NAT}(\gamma)} x^T q^{\mathrm{maj}(T)} t^{\mathrm{coinv}(T)} \prod_{\substack{u \in \widehat{\mathrm{dg}}(\mathrm{inc}(\gamma)) \\ T(u) \neq T(\mathrm{South}(u))}} \frac{(1-t)}{(1-q^{\mathrm{leg}(u)+1} t^{\mathrm{arm}(u)+1})}$$

packed fillings

- a packed filling is one that contains all the integers in the set $[1, m]$ where m is the largest entry
- if T' **compresses** to a packed filling T, then $\mathrm{maj}(T') = \mathrm{maj}(T)$, $\mathrm{coinv}(T') = \mathrm{coinv}(T)$, and $x^{T'}$ is a component of $M_T(X)$.
- Thus we compute G_γ as a sum over packed fillings:

Example: $\gamma = (1, 2)$

$M_{\{1\}}$ $\quad \frac{qt(1-t)}{1-qt^2} M_{\{1,2\}}$ $\quad \frac{(1-t)}{1-qt^2} M_{\{1,2\}}$ $\quad \frac{t(1-t)}{1-qt^2} M_{\{1,2\}}$

$$G_{(1,2)} = M_{\{1\}} + \frac{(1-t)(1+t+qt)}{1-qt^2} M_{\{1,2\}}.$$

standard fillings

- a standard filling is one where each integer $[1, n]$ appears exactly once
- the standardization τ of T is a standard filling that satisfies $\mathrm{coinv}(T) = \mathrm{coinv}(\tau)$ and $\mathrm{maj}(T) = \mathrm{maj}(\tau)$

$$T = \begin{array}{ccc} & & \boxed{4} \\ & \boxed{3} & \boxed{1} \\ & \boxed{3} & \boxed{5} \\ \boxed{1} & \boxed{4} & \boxed{2} \end{array} \qquad \tau = \begin{array}{ccc} & & \boxed{6} \\ & \boxed{4} & \boxed{1} \\ & \boxed{5} & \boxed{8} \\ \boxed{2} & \boxed{7} & \boxed{3} \end{array}$$

$\mathrm{ST}(\gamma)$ is the set of standard tableaux obtained from $\mathrm{NAT}(\gamma)$

- restricted cells: $i \in V(\tau)$ if $i + 1$ precedes i in the reading word or if i and $i + 1$ attack each other
- cells with choices: $i \in W(\tau)$ if $i + 1$ is directly below i in τ: $W(\tau) = \{i \in \tau \ : \ \mathrm{South}(i) = i + 1\}$
- For some choice $V(\tau) \subseteq S \subseteq [n - 1]$, the destandardization $T = \delta_S(\tau)$ is a tableau in the pre-image of a standard tableau, where $S \cap W(\tau)$ is some choice of cells c such that that $T(c) < T(\mathrm{South}(c))$.

Example: $W(\tau) = \{4\}$ and $V(\tau) = \{2, 3, 5, 7\}$, and $S = \{2, 3, 5, 7\}$ such that $T = \delta_S(\tau)$

partition of NAT(γ) into standard fillings

The set of packed tableaux NAT(γ) can be partitioned as:

$$\bigsqcup_{\tau \in \mathrm{ST}(\gamma)} \bigcup_{V(\tau) \subseteq S \subseteq [n-1]} \delta_S(\tau).$$

Thus,

$$G_\gamma(X; q, t) = \sum_{\tau \in \mathrm{ST}(\gamma)} \sum_{V(\tau) \subseteq S \subseteq [n-1]} \mathrm{wt}(\delta_S(\tau)) M_S$$

where

$$\mathrm{wt}(\delta_S(\tau)) = t^{\mathrm{coinv}(\tau)} q^{\mathrm{maj}(\tau)} \prod_{\substack{u \in \widehat{\mathrm{dg}}(\gamma) \\ u \notin W(\tau)}} \frac{1-t}{1-q^{\mathrm{leg}(u)+1} t^{\mathrm{arm}(u)+1}}$$

$$\times \prod_{u \in S \cap W(\tau)} \frac{1-t}{1-q^{\mathrm{leg}(u)+1} t^{\mathrm{arm}(u)+1}}$$

fundamental expansion, general case

Theorem (Corteel-M-Roberts '21)

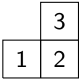
$$G_\gamma(X;q,t) = \sum_{\tau \in \mathrm{ST}(\gamma)} t^{\mathrm{coinv}(\tau)} q^{\mathrm{maj}(\tau)} \left(\prod_{\substack{u \in \widehat{\mathrm{dg}}(\gamma) \\ u \notin W(\tau)}} \frac{1-t}{1-q^{\mathrm{leg}(u)+1} t^{\mathrm{arm}(u)+1}} \right)$$

$$\times \sum_{U \subseteq W(\tau)} (-t)^{|U|} \left(\prod_{u \in U} \frac{1-q^{\mathrm{leg}(u)+1} t^{\mathrm{arm}(u)}}{1-q^{\mathrm{leg}(u)+1} t^{\mathrm{arm}(u)+1}} \right) F_{V(\tau) \cup U}.$$

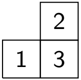
$i \in V(\tau)$ if $i+1$ precedes i in the reading word or if i and $i+1$ attack each other

$$G_{(1,2)} = \frac{qt-t^2}{1-qt^2} F_{\{1,2\}} + F_{\{1\}}$$

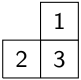
$$= \frac{qt(1-t)}{1-qt^2} F_{\{1,2\}} + 1 \left(F_{\{1\}} + (-t) \frac{1-qt}{1-qt^2} F_{\{1,2\}} \right) + \frac{t(1-t)}{1-qt^2} F_{\{1,2\}}$$



$V(\tau)=\{1,2\}$
$W(\tau)=\emptyset$
$\mathrm{wt}(\tau)=\frac{qt(1-t)}{1-qt^2}$



$V(\tau)=\{1\}$
$W(\tau)=\{2\}$
$\mathrm{wt}(\tau)=1$



$V(\tau)=\{1,2\}$
$W(\tau)=\emptyset$
$\mathrm{wt}(\tau)=\frac{t(1-t)}{1-qt^2}$

further results

- simplification of the quasisymmetric Hall-Littlewood polynomial expansion: $q = 0$

$$G_\gamma(X;0,t) = \sum_{\tau \in \mathsf{ST}_1(\gamma)} (1-t)^{\omega(\tau)}(-t)^{|\operatorname{Des}(\tau)|} t^{\operatorname{coinv}(\tau)-\operatorname{coinv}(\operatorname{Des}(\tau))} F_{V(\tau)}.$$

where $\mathsf{ST}_1(\gamma) = \{\tau \in \mathsf{ST}(\gamma) \colon i \in \operatorname{Des}(\tau) \implies \operatorname{South}(i) = i-1\}$.

- quasisymmetric Jack polynomial expansion: $t = q^\alpha$, $q \to 1$

$$G_\gamma(X;\alpha) = \sum_{\tau \in \mathsf{ST}(\gamma)} \left(\prod_{u \in W(\tau)} (\alpha(\operatorname{leg}(u)+1) + \operatorname{arm}(u) + 1) \right) \times \sum_{U \subseteq W(\tau)} (-1)^{|U|} \left(\prod_{u \in U} \frac{\alpha(\operatorname{leg}(u)+1)+\operatorname{arm}(u)}{\alpha(\operatorname{leg}(u)+1)+\operatorname{arm}(u)+1} \right) F_{V(\tau)\cup U}.$$