Shifted Bender–Knuth moves and a shifted Berenstein–Kirillov group

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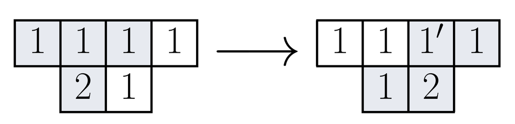
Shifted semistandard tableaux

- A **shifted semistandard tableau** T is a filling of the shifted diagram λ/μ (strict partitions) in the alphabet $[n]' := \{1' < 1 < \cdots < n' < n\}$ such that rows and columns are weakly increasing and there is at most one i' per row and one i per column, for each $i \in [n]$. The set of such tableaux in canonical form (first occurrence of each i or i' must be unprimed [3]) is denoted by $\mathsf{ShST}(\lambda/\mu, n)$.
- The **shifted tableaux switching**, introduced by Choi, Nam and Oh [2] takes a pair of shifted tableaux (S,T), with T extending S, and moves one through the other, in a sequence determined by S, obtaining a pair $({}^{S}T, S_T)$:



The same result may be obtained using the type C infusion due to Thomas and Yong [8] on standardized tableaux followed by the shifted semistandardization due to Pechenik and Yong [5].

Shifted tableau crystals and a cactus group action

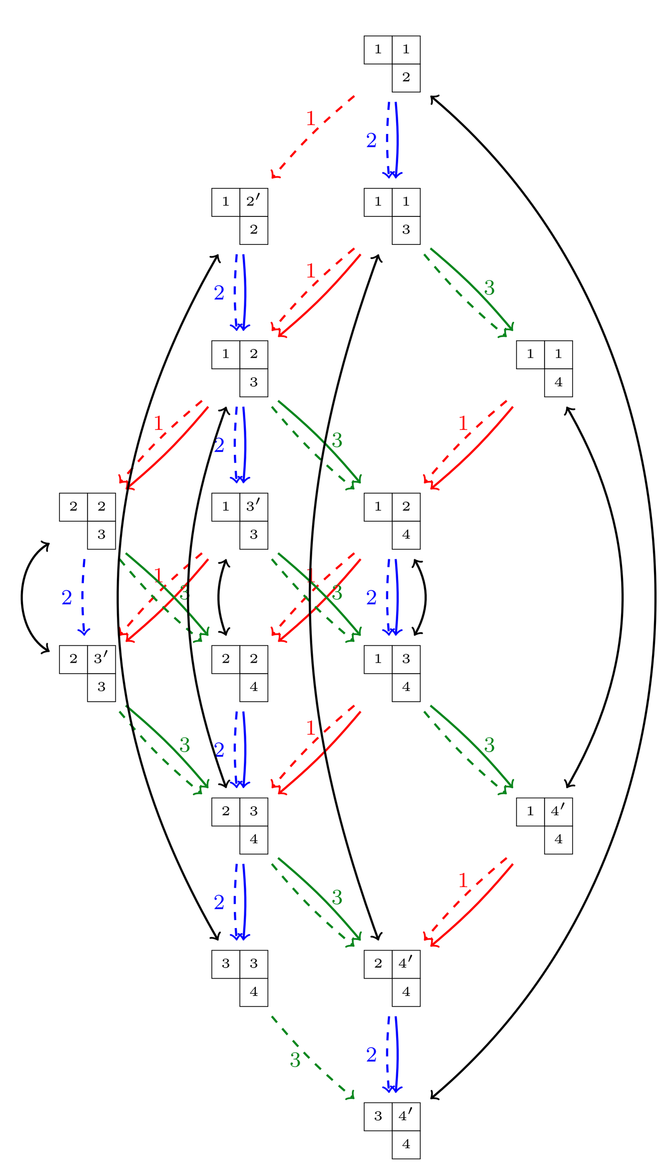
- Gillespie, Levinson, and Purbhoo [3] introduced a crystal-like structure on $\mathsf{ShST}(\lambda/\mu, n)$, defining coplactic primed and unprimed crystal operators E'_i, E_i, F'_i and F_i, length functions ε_i $(\hat{\varepsilon}_i, \varepsilon'_i)$ and φ_i $(\hat{\varphi}_i, \varphi'_i)$ and weight function wt, for $I := [n-1]$.
- This structure admits a decomposition into connected components, each one being isomorphic to some $\mathsf{ShST}(\nu, n)$ via rectification.

The Lusztig–Schützenberger involution

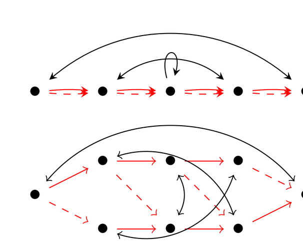
The Lusztig–Schützenberger involution is the unique map of sets η on $\mathsf{ShST}(\nu, n)$, such that, for any $T \in \mathsf{ShST}(\nu, n)$, for $i \in I$:

1. $E'_i\eta(T) = \eta F'_{n-i}(T)$.
2. $E_i\eta(T) = \eta F_{n-i}(T)$.
3. $F'_i\eta(T) = \eta E'_{n-i}(T)$.
4. $F_i\eta(T) = \eta E_{n-i}(T)$.
5. $\mathsf{wt}(\eta(T)) = \mathsf{wt}(T)^{\mathsf{rev}}$.

It is extended by coplacity to the connected components of $\mathsf{ShST}(\lambda/\mu, n)$. It is realized by the **shifted reversal** (or **shifted evacuation** for straight shapes).



- The **partial Lusztig–Schützenberger involution** is the restriction to an interval $[i,j]'$, $\eta_{i,j}(T) = T^{1,i-1} \sqcup \eta(T^{i,j}) \sqcup T^{j+1,n}$.
- The **shifted reflection operators** are $\sigma_i := \eta_{i,i+1}$.
- These operators **do not** need to satisfy the braid relations $(\sigma_i\sigma_{i+1})^3 = 1$, and thus, do not realize an action of $\mathfrak{S}_n$ on $\mathsf{ShST}(\lambda/\mu, n)$.



The n-fruit cactus group

The **n-fruit cactus group** J_n is the free group with generators $s_{i,j}$, for $1 \le i < j \le n$, subject to the relations:

1. $s_{i,j}^2 = 1$.
2. $s_{i,j}s_{k,l} = s_{k,l}s_{i,j}$, for $[i,j] \cap [k,l] = \emptyset$.
3. $s_{i,j}s_{k,l} = s_{i+j-l,i+j-k}s_{i,j}$, for $[k,l] \subseteq [i,j]$.

Theorem (R. 2020)

The cactus group J_n acts naturally on $\mathsf{ShST}(\lambda/\mu, n)$ by $s_{i,j} \mapsto \eta_{i,j}$, for $1 \le i < j \le n$. In particular, it acts on $\mathsf{ShST}(\nu, n)$ by $s_{1,i} \mapsto \mathsf{evac}_i$, where evac_i is the restriction of the shifted evacuation to $[1,i]'$.

The Berenstein–Kirillov group

- The **Berenstein–Kirillov group** $\mathcal{BK}$ is the free group generated by the classic Bender–Knuth involutions, modulo the relations they satisfy on semistandard Young tableaux. We consider its subgroup $\mathcal{BK}_n$ generated by the first $n-1$ Bender–Knuth involutions.

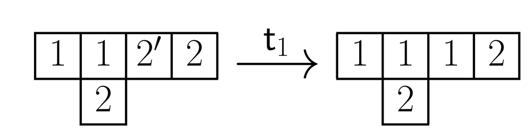
Theorem (Chmutov, Glick, Pylyavskyy 2020, Halacheva 2020)

The map $\varphi : s_{i,j} \mapsto q_{j-1}q_{j-i}q_{j-1}$ is an epimorphism from J_n to $\mathcal{BK}_n$, for $1 \le i < j \le n$, where $q_i := t_1(t_2t_1)\cdots(t_it_{i-1}\cdots t_1)$. Thus $\mathcal{BK}_n$ is isomorphic to $J_n/\ker\varphi$.

- The kernel $\ker\varphi$ is non-trivial, as it contains $\langle \phi^{-1}((t_1t_2)^6)\rangle$.

A shifted Berenstein–Kirillov group

- The shifted Bender–Knuth move t_i is defined on $T \in \mathsf{ShST}(\lambda/\mu, n)$ by applying the shifted tableau switching on the pair (T^i, T^{i+1}), followed by a swapping of the labels $i \leftrightarrow i+1$ and $i' \leftrightarrow (i+1)'$.



- We define the **shifted Berenstein–Kirillov group** $\mathcal{SBK}$ to be the free group generated by the shifted Bender–Knuth involutions t_i, modulo the relations they satisfy on semistandard shifted tableaux, and let $\mathcal{SBK}_n$ be its subroup generated by the first $n-1$ shifted Bender–Knuth involutions.

Properties of t_i

1. $\mathsf{t}_i^2 = 1$, for $i > 1$
2. $\mathsf{t}_i\mathsf{t}_j = \mathsf{t}_j\mathsf{t}_i$, for $|i-j| > 1$.
3. $(\mathsf{t}_i\mathsf{q}_{k-1}\mathsf{q}_{k-j}\mathsf{q}_{k-1})^2 = 1$, for $2 \le i+1 < j < k$, where $\mathsf{q}_i := \mathsf{t}_1(\mathsf{t}_2\mathsf{t}_2)\cdots(\mathsf{t}_i\mathsf{t}_{i-1}\cdots\mathsf{t}_1)$. In particular, $(\mathsf{t}_1\mathsf{q}_\mathsf{i})^4 = 1$, for $i > 2$.

- Unlike the case for the classic Bender–Knuth involutions, the shifted operators **do not** need to satisfy $(\mathsf{t}_1\mathsf{t}_2)^6 = 1$. This relation is equivalent to the braid relations of $\mathfrak{S}_n$.

Theorem (R. 2021)

There is a natural action of $\mathcal{SBK}_n$ on $\mathsf{ShST}(\nu, n)$, given by the group homomorphism $\mathsf{q}_i \mapsto \mathsf{evac}_{i+1}$, which coincides with the previous action of J_n.

Main result (R. 2021)

The map $\psi : s_{i,j} \mapsto \mathsf{q}_{j-1}\mathsf{q}_{j-i}\mathsf{q}_{j-1}$ is an epimorphism from J_n to $\mathcal{SBK}_n$, for $1 \le i < j \le n$. Hence $\mathcal{SBK}_n$ is isomorphic to $J_n/\ker\psi$.

- In general, we do not know if the kernel $\ker\psi$ is non-trivial.

References

[1] M. Chmutov, M. Glick, and P. Pylyavskyy. “The Berenstein–Kirillov group and cactus groups”. *Journal of Combinatorial Algebra* 4.2 (2020), pp. 111–140.

[2] S-I. Choi, S.-Y. Nam, and Y.-T. Oh. “Shifted tableau switchings and shifted Littlewood-Richardson coefficients”. *J. Korean Math. Soc.* 56.4 (2019), pp. 947–984.

[3] M. Gillespie, J. Levinson and K. Purbhoo. “A crystal-like structure on shifted tableau”. *Algebraic Combinatorics* 3.3 (2020), pp. 693–725.

[4] I. Halacheva. “Skew Howe duality for crystals and the cactus group”. 2020. arXiv:2001.02262.

[5] O. Pechenik and A. Yong. “Genomic tableaux”. *J. Algebraic Combin.* 45.3 (2017), pp. 649–685.

[6] I. Rodrigues. “A shifted Berenstein–Kirillov group and the cactus group”. 2021. arXiv:2104.11799.

[7] I. Rodrigues. “An action of the cactus group on shifted tableau crystals”. 2020. arXiv:2007.07078.

[8] H. Thomas and A. Yong. “A combinatorial rule for (co)miniscule Schubert calculus”. *Adv.Math.* 222.2 (2009), pp. 596–620.

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