a mystery group action

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mystery group action

Let $\mathrm{SYT}(\lambda/\mu)$ be the set of standard Young tableaux of shape $\lambda/\mu \vdash r$.

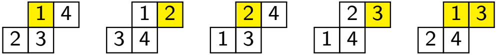
Let $S_{\lambda/\mu} = \bigoplus_\nu S_\nu^{\oplus c_{\mu,\nu}^\lambda}$ be the corresponding representation.

Theorem

$S_{\lambda/\mu} \otimes S_{\lambda/\mu} \downarrow_{\langle (1,\ldots,r)\rangle}$ is isomorphic to a cyclic group action.

Example

Let $\lambda/\mu = (3,2)/(1)$. The character of $S_{\lambda/\mu} \otimes S_{\lambda/\mu} \downarrow_{\langle (1,\ldots,r)\rangle}$ is



$$\Bigg(\overbrace{\sum_{T \in \mathrm{SYT}(\lambda/\mu)} q^{\mathrm{maj}(T)}}^{f^{\lambda/\mu}(q)} \Bigg)^2 = (q + q^2 + q^2 + q^3 + q^4)^2$$

$$\equiv 1 + 6(1 + q + q^2 + q^3) \mod (q^4 - 1)$$

The group action must have one orbit of size 1 and six orbits of size 4.

mystery statistic

Let rot be the rotation of the chord diagram of a permutation.

Theorem

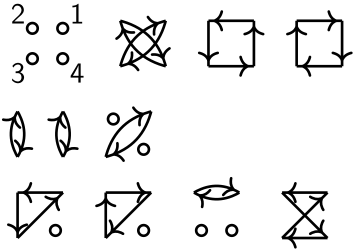
$\exists s : \mathfrak{S}_r \to$ *integer partitions of size r*

- $s \circ \mathrm{rot} = s$
- rot *restricted to* $\mathfrak{S}_r^\lambda = \{\sigma \in \mathfrak{S}_r | s(\sigma) = \lambda\}$ *has character*

  $f^\lambda(q)^2 = \left(\sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}\, T}\right)^2$

(⇒ s is equidistributed with the Robinson-Schensted shape)

Example



	orbit sizes	
(4) :	1	[1,2,3,4]
$(3,1)$:	4, 4, 1	[1,2,4,3],...
(2^2) :	2, 2	[2,1,4,3],...
$(2,1^2)$:	4, 4, 1	[1,4,3,2],...
(1^4) :	1	[4,3,2,1]

mystery basis

Let $\mathfrak{gl}_n \cong V \otimes V^*$ be the adjoint representation of GL_n:

$$\mathrm{GL}_n \to \mathrm{End}(\mathfrak{gl}_n), \quad T \cdot A = TAT^{-1}$$

Let $\mathcal{A}_r^{(n)}$ be the set of $\mathfrak{gl}_n$-highest weight words of weight 0, indexing a basis of $\left(\mathfrak{gl}_n^{\otimes r}\right)^{\mathrm{GL}_n}$.

Let pr be 'promotion' on $\mathcal{A}_r^{(n)}$.

Theorem

There are sets $\mathfrak{S}_r^{(1)} \subseteq \mathfrak{S}_r^{(2)} \subseteq \cdots \subseteq \mathfrak{S}_r^{(r)} = \mathfrak{S}_r$ *and a bijection* $\mathcal{P} : \mathcal{A}_r^{(r)} \to \mathfrak{S}_r^{(r)}$ *with* $\mathcal{P} \circ \mathrm{pr} = \mathrm{rot} \circ \mathcal{P}$ *and* $\mathcal{P}(\mathcal{A}_r^{(n)}) = \mathfrak{S}_r^{(n)}$.

Example

$\mathfrak{S}_r^{(2)}$ is in bijection with non-crossing set-partitions.

For $n \leqslant 2$ and for $n \geqslant r-1$, a variant of the Robinson-Schensted correspondence works.

inequality

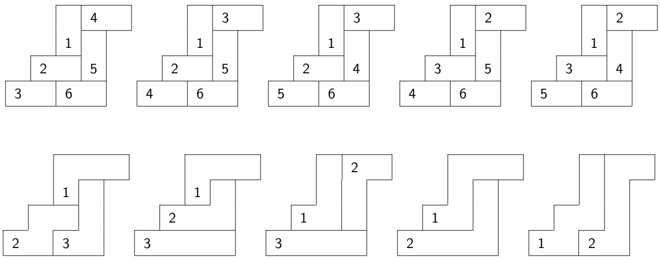
Let $\mathrm{BST}(\lambda/\mu, k)$ be the set of border strip tableaux of shape λ/μ with strips of size k.

Theorem

$$\#\mathrm{BST}(\lambda/\mu, k) \geqslant \sum_{d>1} \#\mathrm{BST}(\lambda/\mu, kd), \text{ if } \#\mathrm{BST}(\lambda/\mu, k) \geqslant 2$$

Example

$\lambda/\mu = (5, 4^3)/(2^2, 1)$, $k = 2$, English notation.



behind the scenes

Proposition (Alexandersson-Amini)

A polynomial $f \in \mathbb{N}[q]$ such that $f(\xi^d) \in \mathbb{N}$ for any $d \in \mathbb{N}$ is the character of a cyclic group action if and only if

$$\mathcal{E}_k = \sum_{d|k} \mu(k/d) f(\xi^d) \geqslant 0 \quad \text{for every } k|r$$

where $\mu(m)$ is the Möbius function:

$$\mu(m) = \begin{cases} (-1)^{\#\text{prime factors of } m} & \text{if } m \text{ is square-free} \\ 0 & \text{otherwise.} \end{cases}$$

In this case, $\mathcal{E}_k$ is the number of elements in orbits of length k.

Proposition

$$f^{\lambda/\mu}(\xi^d)^2 = \#BST(\lambda/\mu, r/d)^2$$