A criterion for sharpness in tree enumeration and the asymptotic number of triangulations in Kuperberg's G_2 spider

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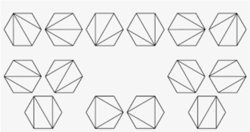
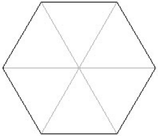
Motivating Problem





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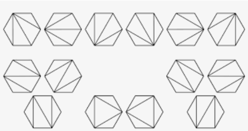

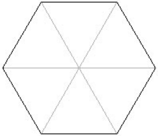

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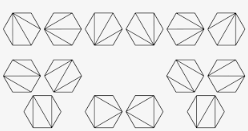
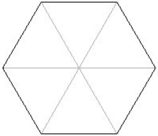

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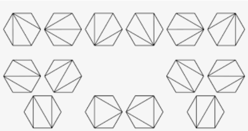 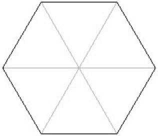

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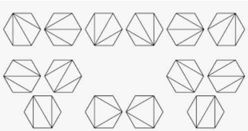

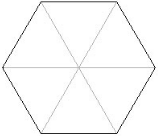

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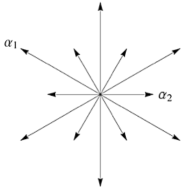
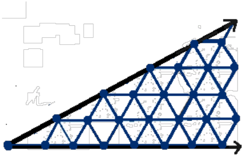
Combinatorial interpretation: 



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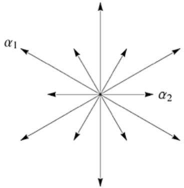

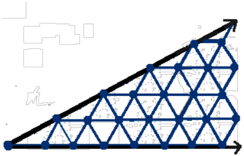

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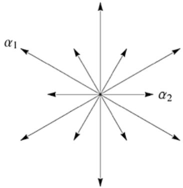

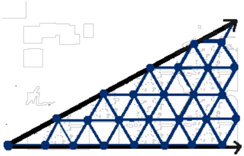

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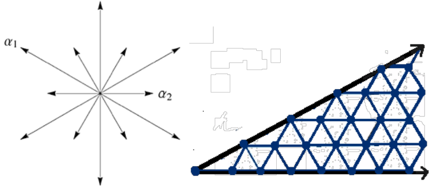

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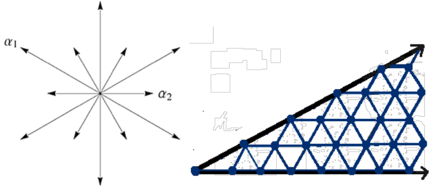

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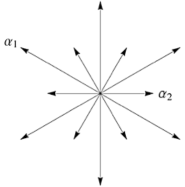

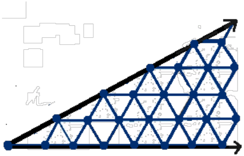

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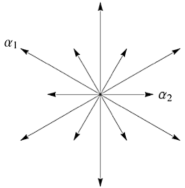

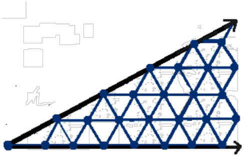

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(b) **Asymptotic formula for a_n.**

$$a_n = M\frac{\rho^n}{n^7}\left(1 + \mathcal{O}\left(\frac{\log n}{n}\right)\right) \quad (\text{as } n \to \infty)$$

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Graph theoretical interpretation of $B(x) = A(xB(x))$:
$b_n = \#$ of rooted planar trees with n vertices such that each vertex having i children can have one of a_i distinct colors.

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Theorem of Meir and Moon (1978): If there exists $\tau \in [0, R)$ where $\psi'(\tau) = 0$, then:

(1) $B(x)$ has radius of convergence $r = \tau / A(\tau)$, and $rB(r) = \tau < R$.

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By contraposition of the theorem,
$b_n \neq \Theta(r^{-n} n^{-3/2}) \implies \psi'(\tau) \neq 0$ for all $\tau \in [0, R)$.
If $0 < rB(r) < R$, then ψ is locally invertible at $rB(r)$, and this local inverse extends B as a complex analytic function beyond its disk of convergence at the point $\psi(rB(r)) = r \in \mathbb{C}$. Contradiction.

Proof of Conjecture

By character theory, Kuperberg worked out that b_n = coefficient of $x^n y^n$ in the Laurent polynomial WM^n, where

$$M(x,y) = 1 + x + y + xy + x^2y + xy^2 + (xy)^2,$$

and

$$\begin{aligned} W(x,y) = x^{-2}y^{-3}(&x^2y^3 - xy^3 + x^{-1}y^2 - x^{-2}y + x^{-3}y^{-1} - x^{-3}y^{-2} \\ &+x^{-2}y^{-3} - x^{-1}y^{-3} + xy^{-2} - x^2y^{-1} + x^3y - x^3y^2). \end{aligned}$$

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By a saddle-point estimate,

$$\begin{aligned} b_n &= \frac{1}{(2\pi i)^2} \oint \oint \left[W(z_1,z_2) \cdot M(z_1,z_2)^n \cdot \frac{1}{(z_1 z_2)^{n+1}} \right] dz_1\, dz_2 \\ &= \frac{4117715\sqrt{3}}{864\pi} \frac{7^n}{n^7} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \end{aligned}$$

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$$B(z) = p(Z) - \frac{K}{6!} Z^6 \log Z + Z^7 H_2(Z) + Z^7 H_1(Z) \log Z\,,$$

where H_1 and H_2 are power series, and p is a degree-6 polynomial.

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Recall:

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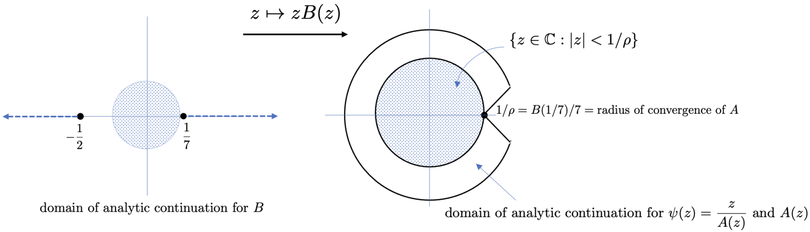

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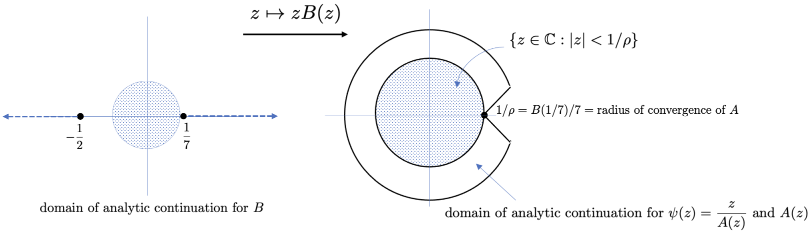

With $V = 1 - \rho z$,

$$A(z) = \eta(V) - \frac{49C}{\rho} V^6 \log(V) + \mathcal{O}(V^7 \log V),$$

as $z \to 1/\rho$, where η is a degree-seven polynomial.

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Theorem of Flajolet and Odlyzko (1990):

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This implies the desired estimate in the Main Result:

$$a_n = M\frac{\rho^n}{n^7} + \mathcal{O}\left(\frac{\log n}{n^8}\right)$$