On Combinatorial Models for Affine Crystals

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Abstract

We give an explicit description of the unique crystal isomorphism between two combinatorial models for tensor products of Kirillov-Reshetikhin crystals: the tableau model and the quantum alcove model.

Crystal Bases

Main idea: use colored directed graphs to encode certain representations V of the quantum group $U_q(\mathfrak{g})$ as $q \to 0$ ($\mathfrak{g}$ complex semisimple or affine Lie algebra).
Kashiwara (crystal) operators are modified versions of the Chevalley generators (indexed by the simple roots α_i): $\tilde{e}_i, \tilde{f}_i$.
V has a *crystal basis* $\boldsymbol{B}$

$$\tilde{e}_i, \tilde{f}_i : \mathbf{B} \to \mathbf{B} \sqcup 0,$$
$$\tilde{f}_i(b) = b' \Leftrightarrow \tilde{e}_i(b') = b$$

Crystal graph: directed graph on $\mathbf{B}$ with edges $b \xrightarrow{i} b'$ exactly when $\tilde{f}_i(b) = b'$.

Kirillov-Reshetikhin (KR) crystals

Correspond to certain *finite*-dimensional representations (not highest weight) of affine Lie algebras $\hat{\mathfrak{g}}$. Consider the untwisted affine types $\mathbf{A}_{n-1}^{(1)} - \mathbf{G}_2^{(1)}$. The corresponding crystals have edges (associated to crystal operators) $\tilde{f}_0, \tilde{f}_1, \ldots$.

Labeled by $p \times q$ rectangles, and denoted $\mathbf{B}^{p,q}$.

Definition. Given a partition $\mathbf{p} = (p_1, p_2, \ldots)$, let

$$\mathbf{B}^{\mathbf{p}} = \mathbf{B}^{p_1,1} \otimes \mathbf{B}^{p_2,1} \otimes \ldots .$$

The crystal operators are defined on $\mathbf{B}^{\mathbf{p}}$ by a tensor product rule.

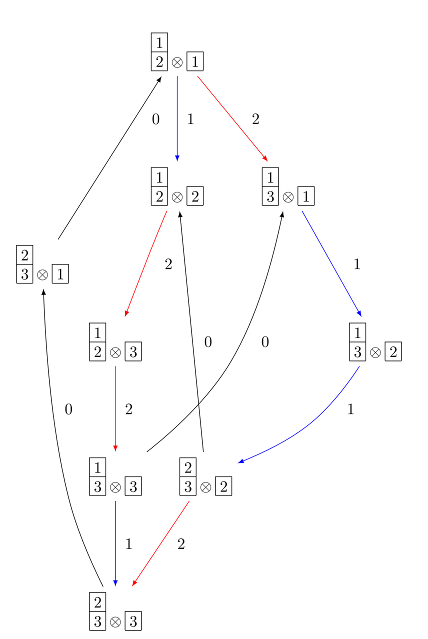
The Tableau Model

With the removal of the $\tilde{f}_0$ arrows, in types A_{n-1} and C_n, we have $\mathbf{B}^{k,1} \cong \mathbf{B}(\omega_k)$ and in types B_n and D_n, we have

$$\mathbf{B}^{k,1} \cong \mathbf{B}(\omega_k) \sqcup \mathbf{B}(\omega_{k-2}) \sqcup \mathbf{B}(\omega_{k-4}) \sqcup \ldots$$

where each $B(\omega_k)$ is given by KN columns of height k. These are strictly increasing fillings of the columns with entries $1 < 2 < \ldots < n$ in type A_{n-1}. With some additional conditions, they are fillings with entries $1 < \ldots < n < \overline{n} < \ldots < \overline{1}$ in type C_n. Types B_n and D_n are similar.

Type A_3 Crystal Graph of $\mathbf{B}^{2,1} \otimes \mathbf{B}^{1,1}$



The Quantum Alcove Model for $\mathbf{B}^{\mathbf{p}}$

The main ingredient is the Weyl group $\mathbf{W} = \langle s_\alpha : \alpha \in \Phi \rangle$.
The *quantum Bruhat graph* on $\mathbf{W}$ is the directed graph with labeled edges $w \xrightarrow{\alpha} ws_\alpha$, where
$l(ws_\alpha) = l(w) + 1$ (Bruhat graph), or
$l(ws_\alpha) = l(w) + 1 - 2\langle \rho, \alpha^\vee \rangle$.

Definition. Given a dominant weight $\lambda = \omega_{p_1} + \ldots + \omega_{p_r}$, we associate with it a sequence of roots, called a $\lambda-chain$ (many choices possible):

$$\Gamma = (\beta_1, \beta_2, \ldots, \beta_m).$$

Let $r_i := s_{\beta_i}$. We consider subsets of positions in Γ

$$J = (j_1 < j_2 < \ldots < j_s) \subseteq \{1, \ldots, m\}.$$

Definition. A subset $J = \{j_1 < j_2 < \ldots < j_s\}$ is *admissible* if we have a path in the quantum Bruhat graph

$$Id = w_0 \xrightarrow{\beta_{j_1}} r_{j_1} \xrightarrow{\beta_{j_2}} r_{j_1} r_{j_2} \ldots \xrightarrow{\beta_{j_s}} r_{j_1} \ldots r_{j_s}.$$

Theorem [LNSSS, 2016]: The collection of all admissible subsets, $\mathcal{A}(\Gamma)$,is a combinatorial model for $\mathbf{B}^{\mathbf{p}}$.

The Two Realizations

- The Tableaux model is simpler and has less structure.
- The Quantum Alcove model has extra structure which makes it easier to do several computations (energy function, combinatorial R-Matrix, keys ...)

Relating the Two Models

We build a forgetful map $fill : \mathcal{A}(\Gamma) \to Tableau(\lambda)$ where $\lambda = \omega_{p_1} + \ldots \omega_{p_r}$.

Definition: For any $k = 1, \ldots, n-1$ we define $\Gamma(k)$ to be the following chain of roots:

$$((k, k+1), (k, k+2), \ldots, (k, n) \ldots$$
$$(2, k+1), (2, k+2), \ldots, (2, n)$$
$$(1, k+1), (1, k+2), \ldots, (1, n))$$

Definition: We construct a λ-*chain* as a concatenation $\Gamma := \Gamma^{\mu_1} \ldots \Gamma^1$ where $\Gamma^j = \Gamma(p_j)$.

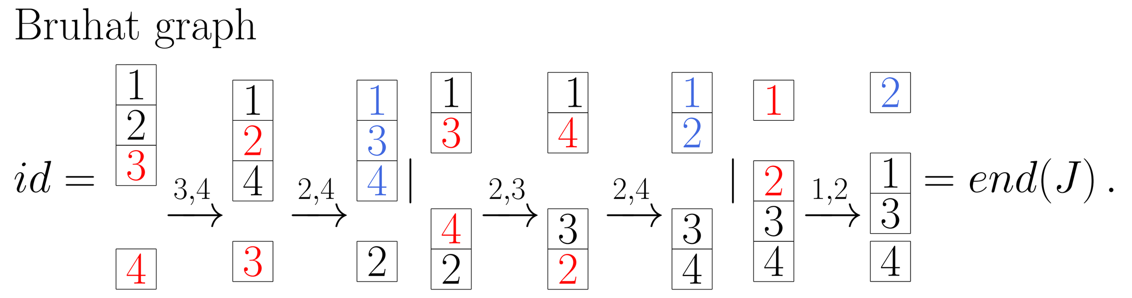
Example Consider $n = 4$ and $\lambda = (3, 2, 1, 0)$. Then the associated λ-chain is $\Gamma = \Gamma^3 \Gamma^2 \Gamma^1 =$

$$((3,4), (2,4), (1,4) | (2,3), (2,4), (1,3), (1,4) | (1,2), (1,3), (1,4)).$$

Example $J = \{1, 2, 4, 5, 8\} \in \mathcal{A}(\Gamma)$.

$$(\underline{(3,4)}, \underline{(2,4)}, (1,4) | \underline{(2,3)}, \underline{(2,4)}, (1,3), (1,4) | \underline{(1,2)}, (1,3), (1,4))$$

We get the corresponding path in the Bruhat order/quantum Bruhat graph



This gives us $fill(J) =$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 & 2 & \\ \cline{1-2} 4 & & \\ \cline{1-1} \end{array} .$$

The Reverse Map in Type A_{n-1}

Consider the vertex in $\bigotimes_{i=1}^{r} B^{p_i,1}$ from the previous example

$$f(T) = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 & 2 & \\ \cline{1-2} 4 & & \\ \cline{1-1} \end{array} .$$

Use entries of columns i and $i-1$ viewed as sets to build the desired sub-list of Γ^i where the zero column is the size n column of strictly increasing entries.
This is done with two algorithms: Reorder and Path
The resulting bijection is a crystal isomorphism [LL,2015].

The Reorder Rule

First, let us consider the *circular order*

$$a \preceq_a a+1 \preceq_a \ldots \preceq_a n \preceq_a 1 \preceq_a \ldots \preceq_a a-1 .$$

We write all chains in $\preceq_a$ starting with a, so the subscript a can be dropped.
Let C and C' be two columns. We fix the entries in C and wish to reorder those in C'.
For each $1 \leq i \leq \#C'$, we have

$$a_i = C'(i) = min\{C'(l) : i \leq l \leq \#C'\}$$

where the minimum is taken with respect to the circle order on $[n]$ starting with $C(i)$.

Example: If $C = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$ and $C' = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$. Then $reorder_C(C') = \begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline 4 \\ \hline \end{array}$.

The Path Algorithm

We now rebuild the desired subsequence of Γ_i by going through Γ_i root by root.

For root (j_1, j_2) if $C[j_1] \prec C[j_2] \prec \hat{C}'[j_1]$ and $C \xrightarrow{(j_1,j_2)} \hat{C}'$ is in the corresponding QBG, then apply it. Otherwise skip. Continue.

So for our example, we have $\Gamma_1 = ((3,4), (2,4), (1,4))$ and get

$$C = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \xrightarrow{(3,4)} \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline 3 \\ \hline \end{array} \xrightarrow{(2,4)} \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline 2 \\ \hline \end{array}$$

The Type C_n Map

- The filling map is similar.
- The inverse map has one major change. Many KN columns have both i and $\overline{i}$ in them, so we use the splitting algorithm [Lecouvey] to bijectively make two columns with no $i, \overline{i}$ pairs in either.
- Then similar Reorder and Path algorithms work.
- So now the reverse map is made up of a process of Split, Reorder, and Path.
- Example:

$$\begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline \overline{5} \\ \hline \overline{4} \\ \hline \overline{3} \\ \hline \end{array} \xrightarrow{split} \begin{array}{|c|c|}\hline 4 & 1 \\ \hline 5 & 2 \\ \hline \overline{3} & \overline{5} \\ \hline \overline{2} & \overline{4} \\ \hline \overline{1} & \overline{3} \\ \hline \end{array}$$

The $\Gamma(k)$ in type C_n comes in two parts. We use the first to get a chain from the left split to the reordered right split and the second to get a chain from the right split to the next column's left split.

The Type B_n Map

- There is a similar filling map
- For the reverse, similar to C_n, we need a splitting map.
- Recall that we now have columns of length $k - 2l$, so we need to Extend back to length k [Briggs].
- Further, the Path algorithm and Reorder algorithms no longer work.
- There is a configuration of two columns CC' that we call being blocked-off.
- Modify Path and Reorder to avoid block-off pattern.

Definition: We say that columns $C = (l_1, l_2, ..., l_k)$, $C' = (r_1, r_2, ..., r_k)$ are *blocked off at* i *by* $b = r_i$
if $|l_i| \leq b$
and$\{1, 2, ..., b\} \subset \{|l_1|, |l_2|, ..., |l_i|\}$
and $\{1, 2, ..., b\} \subset \{|r_1|, |r_2|, ..., |r_i|\}$
and $|\{j : 1 \leq j \leq i, l_j < 0, r_j > 0\}|$ is odd.

The Type D_n Map

The map in type D_n similar to type B_n, but there is a second pattern to be avoided in Reorder and Path.

Bibliography

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