Refined dual Grothendieck polynomials, integrability, and the Schur measure

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Joint work with Kohei Motegi
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Osaka City University

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Table of Contents

Outline

1. **Background**
   - Particle process
   - Schubert Kalculus
2. Results

Definition

Totally

Definition

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Asymmetric

Definition

Totally

Asymmetric

Simple

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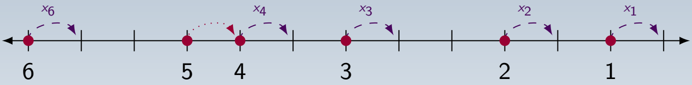

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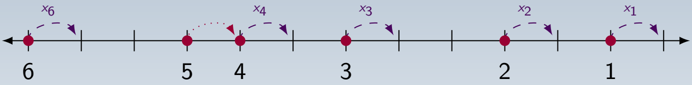

Step initial condition:

Partitions

The positions of the particles correspond to the 01-sequence (a.k.a. Maya diagram) of a partition:

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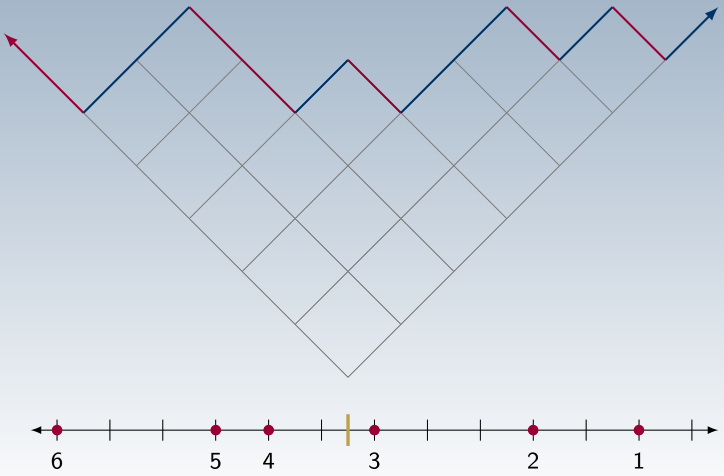

Recording passage times

- Let $G^*(i,j)$ denote time for i-th particle to move j steps.

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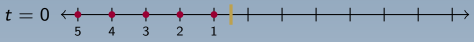
- We use $\mathbb{Z}^2$ coordinates for the matrix, so bottom left corner is w_{11}.

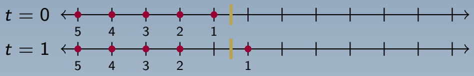
Example



$$G^*(\ell, n) = G(\ell, n) + \ell + n - 1$$

$$G(\ell, n) = \max_{\pi : (1,1) \to (\ell, n)} \sum_{(i,j) \in \pi} w_{ij}.$$

Example



$$\begin{bmatrix} \\ \\ {\color{purple}0}^{1}_{0} & & & \end{bmatrix}$$

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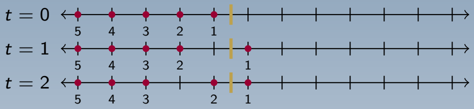

$$\begin{bmatrix} & & & \\ & & & \\ 0^1_0 & 0^2_0 & & \end{bmatrix}$$

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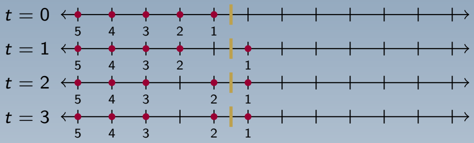

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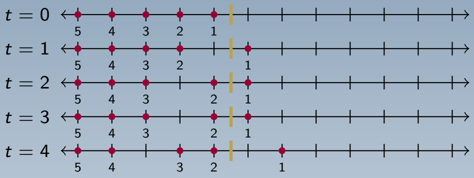

$$\begin{bmatrix} & & & \\ & & & \\ 2_2^4 & & & \\ 0_0^1 & 0_0^2 & 1_1^4 & \end{bmatrix}$$

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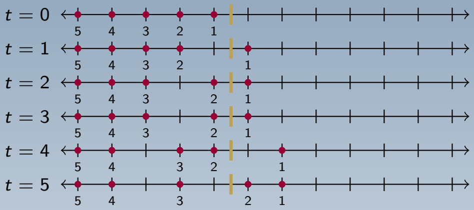

$$\begin{bmatrix} & & & \\ & & & \\ 2^4_2 & 0^5_2 & & \\ 0^1_0 & 0^2_0 & 1^4_1 & \end{bmatrix}$$

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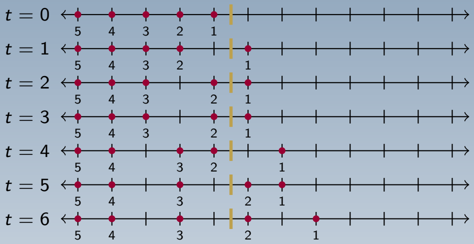

$$\begin{bmatrix} 1_3^6 & & & \\ 2_2^4 & 0_2^5 & & \\ 0_0^1 & 0_0^2 & 1_1^4 & \end{bmatrix}$$

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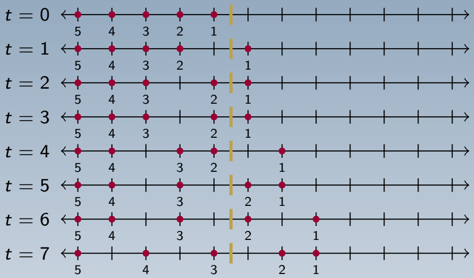

$$\begin{bmatrix} & & & \\ 1^6_3 & 0^7_3 & & \\ 2^4_2 & 0^5_2 & 1^7_3 & \\ 0^1_0 & 0^2_0 & 1^4_1 & 2^7_3 \end{bmatrix}$$

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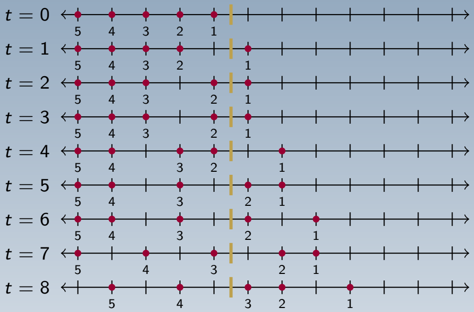

$$\begin{bmatrix} 1^8_4 & & & & \\ 1^6_3 & 0^7_3 & 0^8_3 & & \\ 2^4_2 & 0^5_2 & 1^7_3 & 0^8_3 & \\ 0^1_0 & 0^2_0 & 1^4_1 & 2^7_3 & 0^8_3 \end{bmatrix}$$

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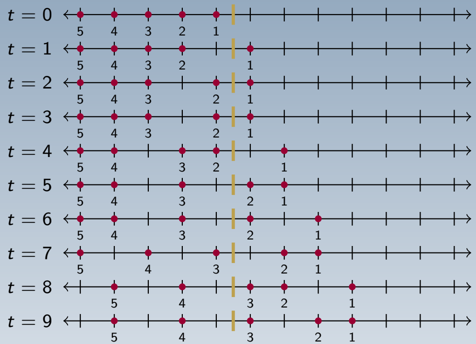

$$\begin{bmatrix} 1^8_4 & 0^9_4 & & & \\ 1^6_3 & 0^7_3 & 0^8_3 & & \\ 2^4_2 & 0^5_2 & 1^7_3 & 0^8_3 & \\ 0^1_0 & 0^2_0 & 1^4_1 & 2^7_3 & 0^8_3 \end{bmatrix}$$

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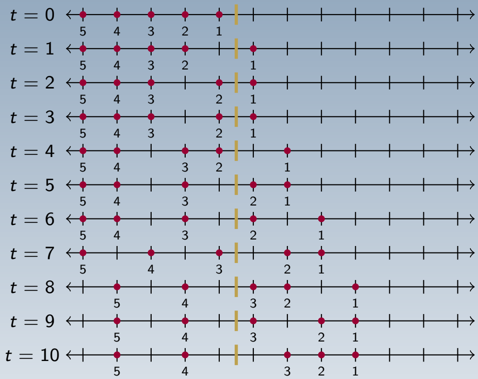

$$\begin{bmatrix} 1^{8}_{4} & 0^{9}_{4} & 0^{10}_{4} & & \\ 1^{6}_{3} & 0^{7}_{3} & 0^{8}_{3} & & \\ 2^{4}_{2} & 0^{5}_{2} & 1^{7}_{3} & 0^{8}_{3} & \\ 0^{1}_{0} & 0^{2}_{0} & 1^{4}_{1} & 2^{7}_{3} & 0^{8}_{3} \end{bmatrix}$$

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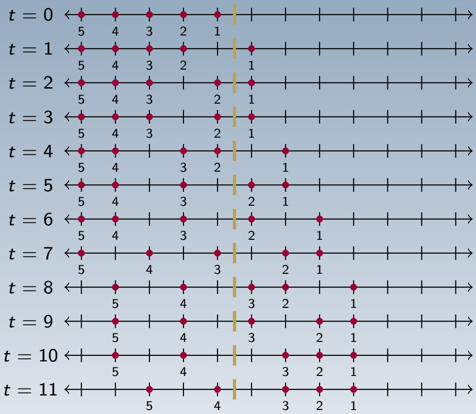

$$\begin{bmatrix} 1^{8}_{4} & 0^{9}_{4} & 0^{10}_{4} & & \\ 1^{6}_{3} & 0^{7}_{3} & 0^{8}_{3} & 2^{11}_{5} & \\ 2^{4}_{2} & 0^{5}_{2} & 1^{7}_{3} & 0^{8}_{3} & 2^{11}_{5} \\ 0^{1}_{0} & 0^{2}_{0} & 1^{4}_{1} & 2^{7}_{3} & 0^{8}_{3} \end{bmatrix}$$

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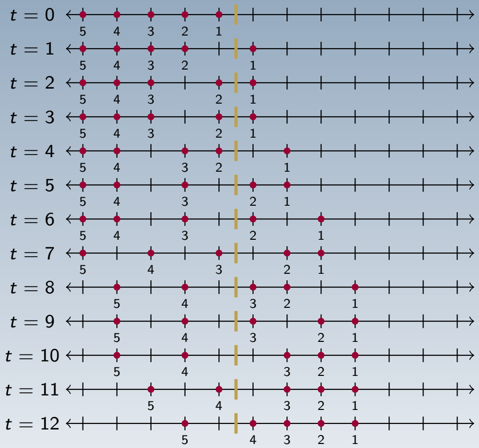

$$\begin{bmatrix} 1_4^8 & 0_4^9 & 0_4^{10} & 0_5^{12} & \\ 1_3^6 & 0_3^7 & 0_3^8 & 2_5^{11} & 0_5^{12} \\ 2_2^4 & 0_2^5 & 1_3^7 & 0_3^8 & 2_5^{11} \\ 0_0^1 & 0_0^2 & 1_1^4 & 2_3^7 & 0_3^8 \end{bmatrix}$$

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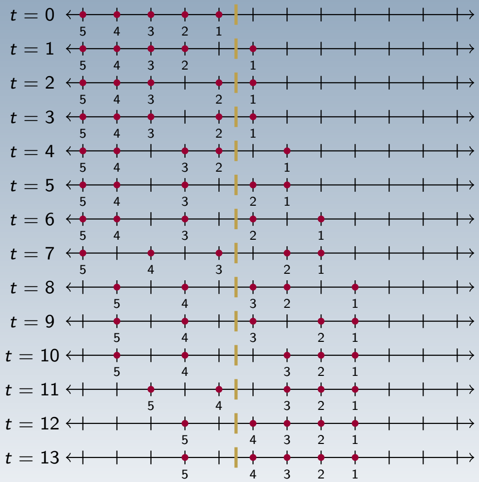

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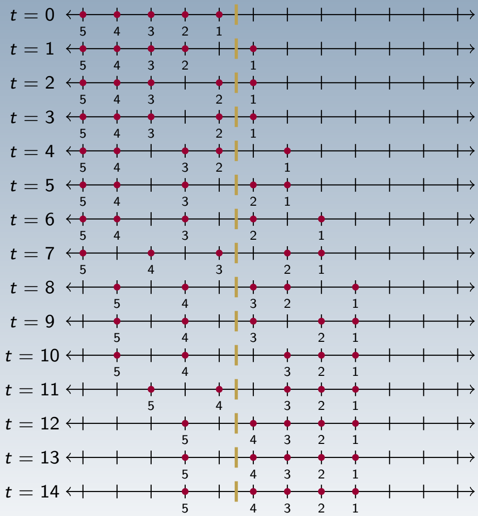

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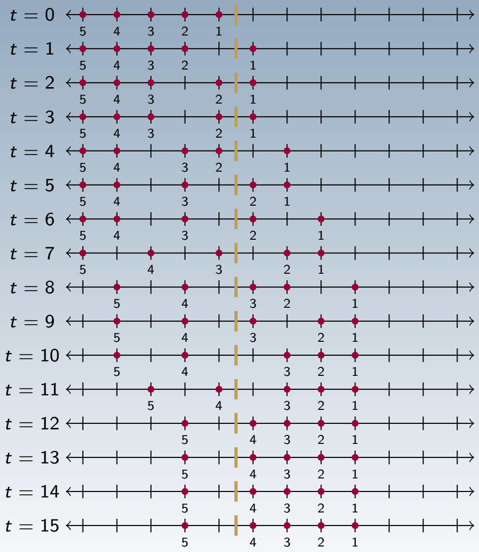

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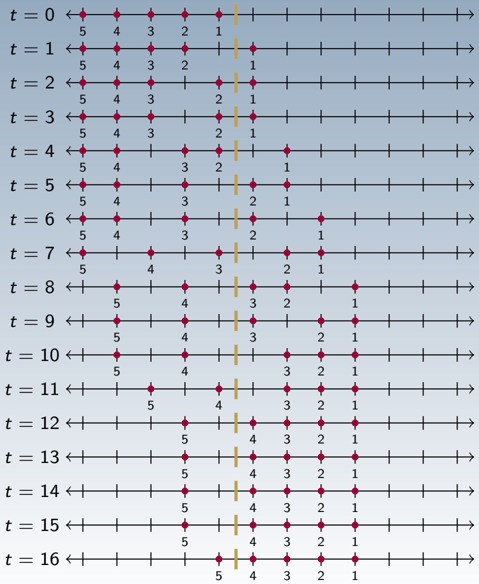

$$\begin{bmatrix} 1_4^8 & 0_4^9 & 0_4^{10} & 0_5^{12} & 3_8^{16} \\ 1_3^6 & 0_3^7 & 0_3^8 & 2_5^{11} & 0_5^{12} \\ 2_2^4 & 0_2^5 & 1_3^7 & 0_3^8 & 2_5^{11} \\ 0_0^1 & 0_0^2 & 1_1^4 & 2_3^7 & 0_3^8 \end{bmatrix}$$

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Refined dual Grothendieck polynomials

- A *reverse plane partition (RPP)* of shape λ is a filling of λ by positive integers such that rows and columns are weakly increasing.

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Definition (Galashin–Grinberg–Liu, 2016)

$$g_\lambda(\mathbf{x};\mathbf{t}) := \sum_{R \in \mathrm{RPP}_\lambda} t_1^{a_1} \cdots t_{\ell-1}^{a_{\ell-1}} x_1^{c_1} \cdots x_n^{c_n},$$

where a_j are the number of boxes in row j containing an i with an i in the box immediately below and c_i is the number of columns containing an i.

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Theorem (Lam–Pylyavskyy, 2008)

$\{g_\lambda(\mathbf{x};\beta)\}_\lambda$ is the dual basis under the Hall inner product (defined as $\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}$) to the β-Grothendieck polynomials that represent Schubert classes in the K-theory of the Grassmannian.

Example: $\mathfrak{g}_{222}(x_1, x_2, x_3; \mathbf{t})$

Example

1	1
2	2
3	3

$x_1^2 x_2^2 x_3^2$

1	1
2	2
2	3

$t_2 x_1^2 x_2^2 x_3$

1	1
2	3
3	3

$t_2 x_1^2 x_2 x_3^2$

1	2
2	3
3	3

$t_2 x_1 x_2^2 x_3^2$

1	1
1	2
2	3

$t_1 x_1^2 x_2^2 x_3$

1	1
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$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 3&3\\\hline\end{array}\quad x_1^2x_2^2x_3^2$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 2&3\\\hline\end{array}\quad t_2x_1^2x_2^2x_3$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline 3&3\\\hline\end{array}\quad t_2x_1^2x_2x_3^2$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline 3&3\\\hline\end{array}\quad t_2x_1x_2^2x_3^2$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 1&2\\\hline 2&3\\\hline\end{array}\quad t_1x_1^2x_2^2x_3$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 1&2\\\hline 3&3\\\hline\end{array}\quad t_1x_1^2x_2x_3^2$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 2&2\\\hline 3&3\\\hline\end{array}\quad t_1x_1x_2^2x_3^2$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 1&2\\\hline 1&3\\\hline\end{array}\quad t_1t_2x_1^2x_2x_3$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 1&1\\\hline 2&2\\\hline\end{array}\quad t_1^2x_1^2x_2^2$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 1&2\\\hline 2&2\\\hline\end{array}\quad t_1t_2x_1^2x_2^2$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 2&2\\\hline\end{array}\quad t_2^2x_1^2x_2^2$$

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$x_1^2x_2^2x_3^2$	$t_2x_1^2x_2^2x_3$	$t_2x_1^2x_2x_3^2$	$t_2x_1x_2^2x_3^2$	$t_1x_1^2x_2^2x_3$	$t_1x_1^2x_2x_3^2$	$t_1x_1x_2^2x_3^2$
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$t_1t_2x_1^2x_2x_3$	$t_1^2x_1^2x_2^2$	$t_1t_2x_1^2x_2^2$	$t_2^2x_1^2x_2^2$	$t_1^2t_2x_1^2x_2$	$t_1t_2^2x_1^2x_2$	$t_1^2t_2^2x_1^2$

There are 50 reverse plane partitions in total.

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1	1
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1	2
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1	1
1	2
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$t_1 t_2 x_1^2 x_2 x_3$

1	1
1	1
2	2

$t_1^2 x_1^2 x_2^2$

1	1
1	2
2	2

$t_1 t_2 x_1^2 x_2^2$

1	1
2	2
2	2

$t_2^2 x_1^2 x_2^2$

1	1
1	1
1	2

$t_1^2 t_2 x_1^2 x_2$

1	1
1	2
1	2

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1	1
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$t_1^2 t_2^2 x_1^2$

There are 50 reverse plane partitions in total.

$$g_{222}(\mathbf{x}; \mathbf{t}) = s_{222} + (t_1 + t_2)s_{221} + t_1 t_2 s_{211} + (t_1^2 + t_1 t_2 + t_2^2)s_{22} + t_1 t_2(t_1 + t_2)s_{21} + t_1^2 t_2^2 s_2.$$

Outline

1. Background
2. Results
   - Transition kernel
   - Consequences

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Theorem (Yeliussizov, 2020; Motegi–S., 2020)

There exists a bijection between states of the LPP with $\mathbf{G}(n) = \lambda$ *and* RPP_λ *such that*

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Example

$$\begin{bmatrix} 1_4 & 0_4 & 0_4 & 0_5 & 3_8 \\ 1_3 & 0_3 & 0_3 & 2_5 & 0_5 \\ 2_2 & 0_2 & 1_3 & 0_3 & 2_5 \\ 0_0 & 0_0 & 1_1 & 2_3 & 0_3 \end{bmatrix} \longleftrightarrow$$

1	1	1	1	4	5	5	5
1	1	1	4	4			
1	1	3	5	5			
3	4	4					

Skew refined dual Grothendiecks

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Proposition (Motegi–S., 2020)

These two definitions of $g_{\lambda/\mu}(\mathbf{x}; \mathbf{t})$ agree.

Jacobi–Trudi formula

Theorem (Motegi–S., 2020)

$$g_{\lambda/\mu}(\mathbf{x};\mathbf{t}) = \det\left[\sum_{m=0}^{\infty} \alpha_m^{ij}(\mathbf{t}) h_{\lambda_i-\mu_j-i+j-m}(\mathbf{x})\right]_{i,j=1}^{\ell},$$

where $\alpha_m^{ij}(\mathbf{t}) = \begin{cases} h_m(t_j,\dots,t_{i-1}) & \text{if } i \geq j, \\ e_m(-t_i,\dots,-t_{j-1}) & \text{if } i < j, \end{cases}$

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Example

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3_8 \\ 1 & 0 & 0 & 2 & 0_5 \\ 2 & 0 & 1 & 0 & 2_5 \\ 0 & 0 & 1 & 2 & 0_3 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\ 3 & 4 & 4 & 1 & 1 & 3 & 5 & 5 & 1 & 4 & 4 & 1 & 5 & 5 & 5 \end{bmatrix}$$

$$Q = \begin{array}{ccccccc} 8 & & 5 & & 2 & & 0 \\ & 5 & & 5 & & 1 & \\ & & 5 & & 3 & & \\ & & & 3 & & & \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 4 & 5 & 5 & 5 \\ \hline 3 & 3 & 4 & 4 & 5 \\ \cline{1-5} 4 & 5 & 1 & 1 & 1 \\ \cline{1-5} 1 & 2 & 2 \\ \cline{1-3} \end{array} \longleftrightarrow \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 4 & 5 & 5 & 5 \\ \hline 1 & 1 & 1 & 4 & 4 \\ \cline{1-5} 1 & 1 & 3 & 5 & 5 \\ \cline{1-5} 3 & 4 & 4 \\ \cline{1-3} \end{array}$$

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Corollary (Motegi–S., 2020)

$$g_\lambda(\mathbf{x},\gamma;\mathbf{t}) = \sum_{\mu\subseteq\lambda} \gamma^{\lambda_1-\mu_1} t_1^{\lambda_2-\mu_2} \cdots t_{\ell-1}^{\lambda_\ell-\mu_\ell} g_\mu(\mathbf{x};\gamma,\mathbf{t})$$

Further results

Corollary (Motegi–S., 2020)

$$\prod_{i=1}^{\ell}\prod_{j=1}^{n}(1-t_i x_j)^{-1} = \sum_{\ell(\lambda)\le\ell} \mathbf{t}^{\lambda} g_{\lambda}(\mathbf{x};\mathbf{t}^{-1})$$

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Proposition (Motegi–S., 2020)

$$s_{m^{\ell}}(\mathbf{x},\mathbf{t},\mathbf{y}) = \sum_{\lambda\subseteq m^{\ell}} g_{\lambda}(\mathbf{x};\mathbf{t}) g_{\lambda^{\dagger}}(\mathbf{y};\mathbf{t}^{\dagger}),$$

where $\mathbf{t}^{\dagger} = (t_{\ell-1},\dots,t_1)$ *and* $\lambda^{\dagger}$ *is the complement in* $m\times\ell$ *rectangle.*

Integral formula

Theorem (Motegi–S., 2020)

$$g_{\lambda/\mu}(\mathbf{x};\mathbf{t}) = \det[F_{ij}]_{i,j=1}^{\ell}, \quad \textit{where}$$

$$F_{ij} = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\gamma_r} \dfrac{\prod_{m=i}^{j-1}(1-t_m z)}{\prod_{m=1}^{n}(1-x_m z) z^{\lambda_i-\mu_j+j-i+1}}\, dz & \textit{if } j \geq i, \\ \dfrac{1}{2\pi i}\displaystyle\int_{\gamma_r} \dfrac{1}{\prod_{m=1}^{n}(1-x_m z)\prod_{m=j}^{i-1}(1-t_m z) z^{\lambda_i-\mu_j+j-i+1}}\, dz & \textit{if } j < i. \end{cases}$$

Moreover, when $\mu = \emptyset$, *we have*

$$g_{\lambda}(\mathbf{x};\mathbf{t}) = \frac{1}{(2\pi i)^{\ell}} \oint \cdots \oint \frac{\prod_{i=1}^{\ell} z_i^{\lambda_i+\ell-i} \prod_{1\leq i<j\leq \ell}(z_j - z_i) z_j}{\prod_{i=1}^{\ell}\prod_{m=1}^{\ell}(z_i - x_m) \prod_{1\leq i<j\leq \ell}(z_j - t_i)}\, dz_1 \cdots dz_{\ell}.$$

Thank you!

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