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Suggested Title: Hessenberg Varieties and Combinatorics Edit

January 11, 2022 at 9:23 AM

Hessenberg Varieties and Combinatorics

FPSAC '21

1/11/22

John Shareshian (WUSTL)

Based on joint work with

Michelle Wachs (Miami)

and with

Laura Escobar (WUSTL)

&

Martha Precup

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Definition: A **Hessenberg Vector** is a weakly increasing sequence of integers

$$\underline{m} = (m_1, \dots, m_n)$$

satisfying $j \leq m_j \leq n$ for all j.

Example:

$$(3, 5, 5, 6, 6, 6)$$

Given a Hessenberg vector m and $X \in M_n(\mathbb{C})$

Given a Hessenberg vector $\underline{m}$ and $X \in M_n(\mathbb{C})$, the associated <u>Hessenberg variety</u>, denoted by $\mathcal{B}(X, \underline{m})$, consists of all flags

$$V_\bullet : 0 = V_0 < V_1 < \cdots < V_{n-1} < V_n = \mathbb{C}^n$$

of subspaces of $\mathbb{C}^n$ satisfying

$$X V_j \leq V_{m_j}$$

for all $j \in [n]$.

So, $\mathcal{B}(X, \underline{m})$ is a subvariety of the variety $\mathcal{B}$ of all full flags in $\mathbb{C}^n$

(De Mari–Shayman '88)

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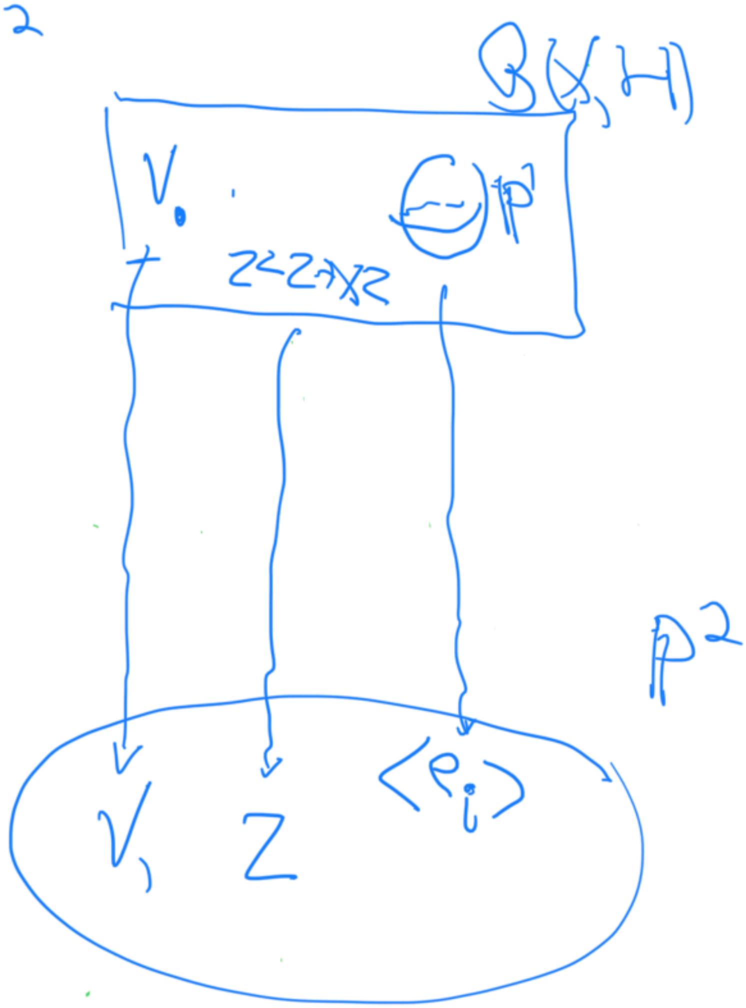
Example: $X = \begin{bmatrix} -1 & & \\ & 0 & \\ & & 1 \end{bmatrix}$ $\underline{m} = (2,3,3)$

- $X \; V_1 < V_2$



$B_0 = 1 \qquad B_2 = 4 \qquad B_4 = 1$

Plan:

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Plan:

① "Old": Regular semisimple Hessenberg varieties and the Stanley-Stembridge Conjecture.

② "New":

(a) Which Schubert varieties are Hessenberg varieties?

(b) Hessenberg varieties of codimension 1 in $\mathcal{B}$

Given a graph $G=(V,E)$, the Chromatic Symmetric Function of G is

$$X_G(x) = \sum \prod$$

Given a graph $G=(V,E)$, the <u>Chromatic Symmetric Function</u> of G is

$$X_G(\underline{x}) = \sum_{\kappa \in K(G)} \prod_{v \in V} x_{\kappa(v)}$$

where

$$\underline{x} := \{x_1, x_2, x_3, \ldots\}$$

$K(G) := \{$ All <u>proper</u> <u>colorings</u> $\kappa : V \to \mathbb{N} \}$

Example

Example: $G =$ 

5 11 5 $\qquad x_5^2 x_{11}$

3 7 6 $\qquad x_3 x_6 x_7$

$$X_G(\underline{x}) = 6m_{111} + m_{21}$$

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Given a Hessenberg vector $\underline{m}$, the associated **Hessenberg graph** is

$$G_{\underline{m}} = ([n], E)$$

where

$$E = \{ ij \mid i < j \le m_i \}$$

Example: $\underline{m} = (3,3,5,5,6,6)$

(Weak) Stanley-Stembridge Conjecture:

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(Weak) Stanley-Stembridge Conjecture:

$X_{G_{\underline{m}}}(\underline{x})$ is e-positive

Example: $\underline{m} = (2,3,3)$ $\quad G_{\underline{m}} = 1 - 2 - 3$

$$X_{G_{\underline{m}}}(\underline{x}) = 6m_{111} + m_{21}$$
$$= 3e_3 + e_{21}$$

Given $G = ([n], E)$ and $\kappa \in K(G)$, the ascent number of κ is

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Given $G=([n],E)$ and $\kappa \in K(G)$, the ascent number of κ is

$$\mathrm{asc}(\kappa) := \#\{ij \in E \mid i<j \text{ and } \kappa(i)<\kappa(j)\}$$

Example:

G

$\mathrm{asc}(\kappa)=2$

κ 7   11   3   28

The chromatic quasisymmetric function of $G=([n],E)$ is

The Chromatic quasisymmetric function of $G=([n],E)$ is

$$X_G(\underline{x};t) := \sum_{\kappa \in K(G)} t^{\operatorname{asc}(\kappa)} \prod_{j=1}^{n} x_{\kappa(j)}$$

$$X_G(\underline{x};t) = \sum t^{\operatorname{asc}(\kappa)} \prod^{n} x_{\kappa(j)}$$

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$$X_G(\underline{x};t) = \sum_{\kappa \in K(G)} t^{\mathrm{asc}(\kappa)} \prod_{j=1}^{n} x_{\kappa(j)}$$

Example: $G = 1 - 2 - 3$

			asc
5	11	5	1
11	5	11	1
3	6	7	2
6	3	7	1
⋮			

$$X_G(\underline{x},t) = (1+4t+t^2)\, m_{111} + t\, m_{21}$$

$$m_{111} + t(4m_{111} + m_{21}) + t^2 m_{111}$$

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<u>Theorem</u> (S-Wachs): If $\underline{m}$ is a Hessenberg vector and

$$X_{G_{\underline{m}}}(x;t) = \sum_j t^j f_j(\underline{x})$$

then each $f_j(\underline{x})$ is a Schur positive symmetric function.

(Proved earlier by Gasharov when $t=1$)

Refined Stanley-Stembridge Conjecture (S-Wachs):

If $\underline{m}$ is a Hessenberg vector and

$$X_{G_{\underline{m}}}(x;t) = \sum_j t^j f_j(\underline{x})$$

then each $f_j(\underline{x})$ is e-positive.

1—2—3

$$X_{G_{\underline{m}}}(\underline{x};t) = m_{111} + t(4m_{111} + m_{21}) + t^2 m_{111}$$

$$e_3 + t(e_{21} + e_3) + t^2 e_3$$

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Let $\underline{m}$ be a Hessenberg vector and assume $X \in M_n(\mathbb{C})$ is regular semisimple.

The torus $T = C_{GL_n(\mathbb{C})}(X)$ acts on $\mathcal{B}(X, \underline{m})$:

$$XV_j \le V_{m_j} \Rightarrow X(YV_j) = Y(XV_j) \le YV_{m_j}$$

for all $Y \in T$.

This action is "equivariantly formal".
GKM (Goresky-Kottwitz-MacPherson) Theory applies

This gives rise to the "dot representation" of

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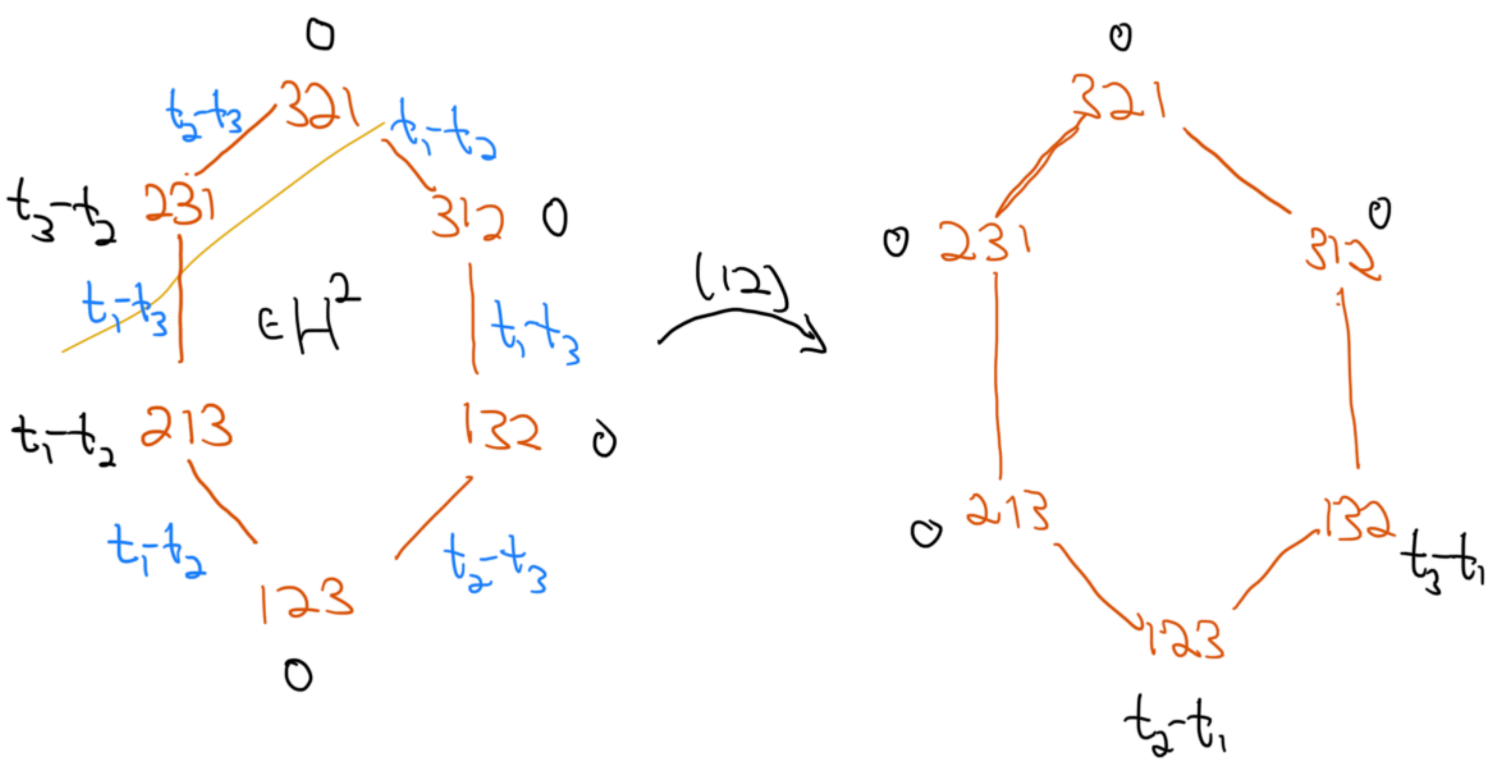
This gives rise to the "dot representation" of S_n on $H^*(\mathcal{B}(X,\underline{m});\mathbb{C})$, studied first by Tymoczko.

Example: $\underline{m} = (2,3,3)$



The Frobenius characteristic assigns to each

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The Frobenius characteristic assigns to each representation of S_n a degree n homogeneous symmetric function

<u>Theorem</u> (Brosnan-Chow): If $\underline{m}=(m_1,\dots,m_n)$ is a Hessenberg vector and $X \in M_n(\mathbb{C})$ is regular semisimple, then

$$ch\left(\sum_j t^j H^{2j}(\mathcal{B}(X,\underline{m}))\right) = \omega X_{G_{\underline{m}}}(x;t).$$

(Another proof was given by Guay-Paquet.)

<u>Corollary</u>: The refined Stanley-Stembridge Conjecture

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Corollary: The refined Stanley-Stembridge Conjecture holds $\iff$ the dot representation of S_n on $H^*(\mathcal{B}(X, \underline{m}); \mathbb{C})$ arises from an action on a set in which each point stabilizer is a Young subgroup.

Recent Progress on the Stanley-Stembridge Conjecture

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Recent Progress on the Stanley-Stembridge Conjecture

- Clearman-Hyatt-Shelton-Skandera showed that the refined Stanley-Stembridge Conjecture is a special case of a conjecture of Haiman about Hecke algebra characters.

- Cho-Huh and Harada-Precup proved the refined conjecture holds when $\alpha(G_{\underline{m}})=2$.

- Cho-Hong proved the refined conjecture holds when $\alpha(G_{\underline{m}})=3$.

- Cho-Hong-Lee identified certain cohomology classes and conjectured that these classes generate the desired permutation modules.

Highlights of recent joint work with Escobar and Precup

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Chow-Hong-Lee identified certain cohomology classes and conjectured that these classes generate the desired permutation modules.

Highlights of recent joint work with Escobar and Precup

- Let $w \in S_n$. If the Schubert variety X_w is a Hessenberg variety, then w avoids the pattern 4231.

- There is no necessary and sufficient pattern avoidance condition for X_w to be a Hessenberg variety.

- One can compute the Poincaré polynomial of $B(X_w, m)$ by ...

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- One can compute the Poincaré polynomial of $\mathcal{B}(X, \underline{m})$ by counting points on an associated Hessenberg variety over $\mathbb{F}_p$, for large enough p

- If $\mathcal{B}(X, \underline{m})$ has codimension one in $Flag_n$, then $\underline{m} = (n-1, n, \ldots, n)$ or $X - \lambda I$ has rank 1 for some $\lambda \in \mathbb{C}$

- If X is nilpotent, then the singular locus of $\mathcal{B}(X, (n-1, n, \ldots, n))$ is $\mathcal{B}(X, (1, n-1, \ldots, n-1, n))$

Thank you!!