The Homology Representation of Subword Order

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The subword order poset

A^* is the free monoid of words of finite length in an alphabet A.

Subword order is defined on A^* by setting $u \leq v$ if u is a subword of v :

$u \leq v \iff u$ is obtained by deleting letters of the word v.

$(A^*, \leq)$ is a graded poset with rank function given by the length $|w|$ of a word w, the number of letters in w.

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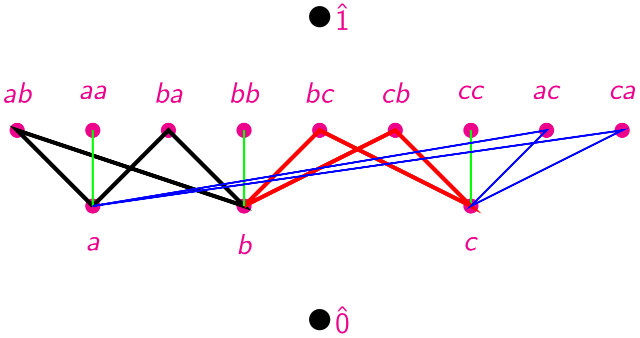
The first two nontrivial ranks of the poset for $|A| = 3$, its order complex and topology

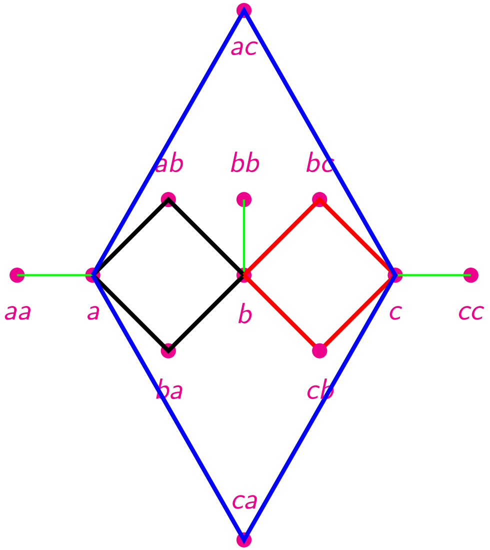
Let $A = \{a, b, c\}$. Consider words of length at most 2 in A.

The least element $\hat{0}$ is the empty word.

There are 3 words of length 1 and 9 words of length 2.



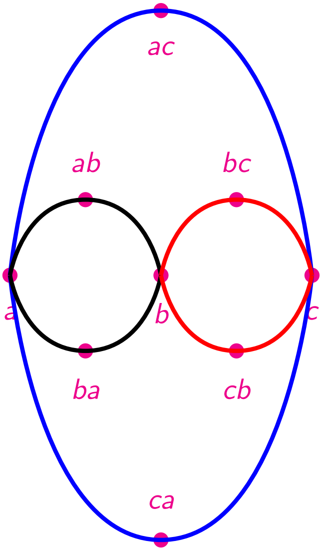
The rank 3 poset for $|A| = 3$, and its 1-dimensional order complex, drawn as a planar graph



The rank-3 poset for $|A| = 3$, and its topology as a wedge of 4 circles

Contract the green edges, and identify a and c with b to get a bouquet of four 1-spheres (circles) with wedge point b.

The homotopy type is that of a wedge of four 1-spheres. Each sphere gives a homology vector space of dimension 1. So the homology is a vector space dimension 4.


ac
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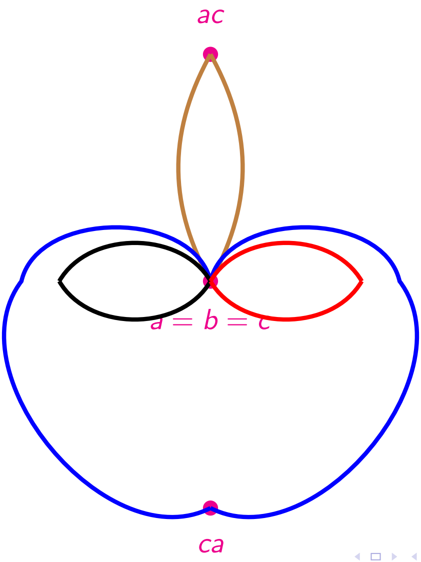
Words of length at most 2 on $A = \{a, b, c\}$: a wedge of four one-spheres



Farmer showed that if $|A| = n$ and $A^*_{n,k}$ is the subposet of A^* consisting of words of length at most k, then $A^*_{n,k}$ has the homology of a wedge of $(n-1)^k$ spheres of dimension $(k-1)$.

Björner showed that the poset $A^*_{n,k}$ is dual-CL shellable, and hence its order complex, as well as the order complex of all rank-selected subposets, is homotopy equivalent to a wedge of spheres.

The action of the symmetric group

Suppose now that the alphabet A is finite, of cardinality n.
The symmetric group S_n acts on $A = \{a_1, \ldots, a_n\}$, and thus on A^* by replacement of letters: $a_i \mapsto a_{\sigma(i)}$ for $\sigma \in S_n$.

Example

$A = \{a_1, a_2, a_3\}$. For $\sigma = (12)$,

$$\sigma \cdot (a_1 a_2 a_1 a_3 a_3 a_2) = a_2 a_1 a_2 a_3 a_3 a_1 .$$

To avoid trivialities we will assume $n \geq 2$.

S_n acts on the chains

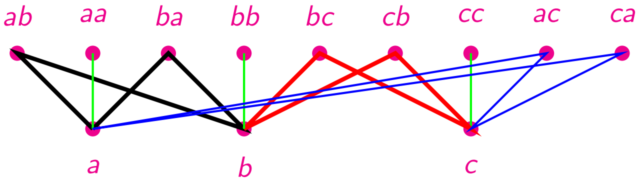
Example (n=3: the 15 chains in $A_{3,2}$, words of length at most 2)

$\{a < ab, b < ab, a < ba, b < ba, a < ac, c < ac, a < ca, c < ca, b < bc, c < bc, b < cb, c < cb\}$ and $\{\mathbf{a} < \mathbf{aa}, \mathbf{b} < \mathbf{bb}, \mathbf{c} < \mathbf{cc}\}$.

Notice:
The three chains $\{\mathbf{a} < \mathbf{aa}, \mathbf{b} < \mathbf{bb}, \mathbf{c} < \mathbf{cc}\}$ span an invariant subspace, closed under the action of S_3. This is the *natural* or *defining* representation V_3 of S_3.

Its S_3-invariant complement W_3 is the 12-dimensional space spanned by the chains of the form $x < xy, x < yx$, where $x \in \{a, b, c\}$ and $y \neq x$.

In this case, the action decomposes into one copy of V_3 and two copies of the regular representation.
This is not obvious from the basis of chains.



The reflection representation

Definition

The natural (or defining) representation V_n of S_n is the action of S_n on the n one-element subsets of $[n]$.

Theorem (Standard Fact)

V_n decomposes into two invariant subspaces; the trivial representation $S_{(n)}$ and the reflection representation $S_{(n-1,1)}$, indexed by the integer partition $(n-1,1)$ of n.

The S_n-action on the homology module $\tilde{H}(A^*_{n,k})$

Theorem (Björner-Stanley)

$\tilde{H}(A^*_{n,k})$ is isomorphic to the kth tensor power of the reflection representation $S_{(n-1,1)}$.

Proof.

(Sketch) Use the Hopf trace formula. The Möbius number calculation can be translated into a character formula for the S_n-action. □

Let S be a subset of the ranks $[1, k]$. Consider the subposet $A^*_{n,k}(S)$ of words with lengths in S. This is also invariant under S_n, and has unique nonvanishing homology (dual CL-shellability is preserved).

Theorem (S, 2020)

For any subset $S = \{1 \leq s_1 < \ldots < s_p \leq k\}$ of $[1, k]$, the action on the chains of $A^*_{n,k}(S)$ is given by the S_n-module

$$\bigotimes_{r=1}^{p} \left(\bigoplus_{i=0}^{s_r - s_{r-1}} \binom{s_r}{i} S_{(n-1,1)}^{\otimes i} \right), s_0 = 1.$$

In particular, it is a nonnegative integer combination of tensor powers of the reflection representation.

The maximal chains

Theorem (S, 2020)

The action of S_n on the maximal chains of $A^*_{n,k}$ decomposes into the sum

$$\bigoplus_{j=1}^{k+1} c(k+1,j) S_{(n-1,1)}^{\otimes k+1-j},$$

where $c(k+1,j)$ is the number of permutations in S_{k+1} with exactly j cycles in its disjoint cycle decomposition.

The dimension version of this is due to Viennot (JCTA, 1983).

Stanley's theory of rank-selected poset homology (JCTA, 1982):

Theorem (Stanley)

Let P be a bounded ranked Cohen-Macaulay poset with automorphism group G, and let S be any subset of ranks. Let P_S be the corresponding rank-selected subposet of P. Let $\alpha_G(S)$, $\beta_G(S)$ denote respectively the actions of G on the maximal chains and the homology of P_S. Then

$$\alpha_G(T) = \sum_{S \subseteq T} \beta_G(S) \quad \text{and thus} \quad \beta_G(T) = \sum_{S \subseteq T} (-1)^{|T|-|S|} \alpha_G(S).$$

Theorem on homology

Theorem (S, 2020)

The S_n-action on the homology of the rank-selected subposet $A^*_{n,k}(T)$, $T \neq \emptyset$, is an integer combination of positive tensor powers of the irreducible indexed by $(n-1,1)$. The highest tensor power that can occur is the $m\mathrm{th}$, where $m = \max(T)$.

Conjecture (A)

Let A be an alphabet of size $n \geq 2$. Then the S_n-action on the homology of any finite nonempty rank-selected subposet of subword order on A^* is a **nonnegative** *integer combination of positive tensor powers of the irreducible indexed by the partition $(n-1,1)$.*

Words of bounded length; the rank-set $[r, k]$

Theorem (S, 2020)

Fix $k \geq 1$ and let S be the interval of consecutive ranks $[r, k]$ for $1 \leq r \leq k$. Then the rank-selected subposet $A^*_{n,k}(S)$ has unique nonvanishing homology in degree $k - r$, and the S_n-homology representation on $\tilde{H}_{k-r}(A^*_{n,k}(S))$ is given by the decomposition

$$\bigoplus_{i=1+k-r}^{k} b_i \, S_{(n-1,1)}^{\otimes i}, \text{ where } b_i = \binom{k}{i}\binom{i-1}{k-r}, i = 1+k-r, \ldots, k.$$

Deleting one rank

Let S be the rank-set $S = [1, k] \backslash \{r\}$, corresponding to the subposet obtained by removing all words of length r, for a fixed r in $[1, k]$.

Theorem (S, 2020)

As an S_n-module, we have

$$\tilde{H}_{k-2}(A^*_{n,k}(S)) \simeq \left[\binom{k}{r} - 1\right] S^{\otimes k}_{(n-1,1)} \oplus \binom{k}{r} S^{\otimes k-1}_{(n-1,1)}.$$

Notice: If $r < k$, the subposet obtained by deleting words of length r has the same homology module as the subposet obtained by deleting words of length $k - r$.

Question: Explicit homotopy equivalence?

Corollary

Let $|A| = n$. Fix a rank $r \in [1, k-1]$. Then the homology modules of the subposets $A^*_{n,k}([1,k]\backslash\{r\})$ and $A^*_{n,k}([1,k]\backslash\{k-r\})$ are S_n-isomorphic.

Question

Is there an S_n-homotopy equivalence between the simplicial complexes associated to the subposets $A^*_{n,k}([1,k]\backslash\{r\})$ and $A^*_{n,k}([1,k]\backslash\{k-r\})$?

Conjecture (A) is true for all rank-selected chain modules, and also the rank-selected homology modules for the rank-set S where

$$(1)\, S = [r, k]; \quad (2)\, S = [1, k] \backslash \{r\}; \quad (3)\, S = \{1 \leq s_1 < s_2\}.$$

Conjecture (A) – (II)

Conjecture (A) is also true for the following:

Theorem (S, 2020)

In the poset $A^*_{n,k}$, for $1 \leq i \leq k$:

1. *The Whitney homology module*

$$WH_i := \bigoplus_{|x|=i} \tilde{H}(\hat{0}, x) \simeq S_{(n-1,1)}^{\otimes i} \oplus S_{(n-1,1)}^{\otimes i-1};$$

2. *The dual Whitney homology module*

$$WH^*_{k+1-i} := \bigoplus_{|x|=k+1-i} \tilde{H}(x, \hat{1}) \simeq \bigoplus_{j=0}^{i} \binom{k}{i} \binom{i}{j} S_{(n-1,1)}^{\otimes j+k-i}.$$

The case $n = 3, k = 2$ revisited

The S_3-module structure on the maximal chains is

$$c(3,1)S_{(n-1,1)}^{\otimes 2} \oplus c(3,2)S_{(n-1,1)} \oplus c(3,3)S_{(3)}.$$

But we know this is a permutation module. In fact, its Frobenius characteristic is (with $*$ denoting the *internal* product):

$$2s_{(2,1)} * s_{(2,1)} + 3s_{(2,1)} + s_{(3)} = h_1 h_2 + 2h_1^3.$$

It is h-positive!

Note: permutation modules are not necessarily h-positive, e.g. S_4 acting on the three set partitions $12/34, 13/24, 14/23$:

$$h_4 + h_2^2 - h_1 h_3.$$

h-positivity

h_n is the complete homogeneous symmetric function of degree n.

Definition (Hooks)

$$\text{Set } T_1(n) := \{h_\lambda : \lambda = (n-r, 1^r), r \geq 1\},$$

$$\text{and } T_2(n) := \{h_\lambda : \lambda = (n-r, 1^r), r \geq 2\}.$$

Theorem (S, 2020)

The Whitney and dual Whitney homology are permutation modules with h-positive Frobenius characteristic supported on the set $T_2(n)$, except for $WH_i, i = 0, 1$.

$\mathrm{ch}\, WH_0 = h_n$, $\mathrm{ch}\, WH_1 = h_1 h_{n-1}$, and for $j \geq 2$,

$$\mathrm{ch}\, WH_j = \sum_{d=2}^{j} S(j-1, d-1)\, h_1^d h_{n-d}.$$

Here $S(n, k)$ is the Stirling number of the second kind.

Action on chains is h-positive

Theorem (S, 2020)

The action of S_n on the maximal chains of the rank-selected subposet of A^* of words with lengths in T, is a **nonnegative** *integer combination of tensor powers of the reflection representation $S_{(n-1,1)}$. The Frobenius characteristic is h-positive and supported on the set $T_1(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geq 1\}$ if $|T| \geq 1$. The coefficient of $h_1 h_{n-1}$ is always 1.*

“almost” h-positivity (I)

Let $s_{(n-1,1)}$ denote the Schur function indexed by the partition $(n-1,1)$.

Theorem (S, 2020)

Let $T \subseteq [1,k]$ be any nonempty subset of ranks in $A^*_{n,k}$. The following statements hold for the Frobenius characteristic $F_n(T)$ of the homology representation $\tilde{H}(A^*_{n,k}(T))$:

1. *its expansion in the basis of homogeneous symmetric functions is an integer combination supported on the set* $T_1(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geq 1\}$.
2. $F_n(T) + (-1)^{|T|} s_{(n-1,1)}$ *is supported on the set* $T_2(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geq 2\}$.

When is this expansion actually h-positive?

Example of "almost" h-positivity: one-rank deletion for $n = 6, k = 7$

Here the rank-set is $T = [1, 7] \setminus \{r\}$.
The reflection representation for S_6 is the irreducible $S_{(5,1)}$, with characteristic $(\mathbf{h_1 h_5 - h_6})$.
Recall that the action of S_6 on the homology is given by

$$\tilde{H}_{k-2}(A^*_{n,k}(T)) \simeq \left[\binom{k}{r} - 1\right] S^{\otimes k}_{(n-1,1)} \oplus \binom{k}{r} S^{\otimes k-1}_{(n-1,1)}.$$

$490h_1^6 + 560h_1^4h_2 + 196h_1^3h_3 + 7h_1^2h_4 - (\mathbf{h_1 h_5 - h_6})$, $(r = 1)$;
$1610h_1^6 + 1820h_1^4h_2 + 630h_1^3h_3 + 21h_1^2h_4 - (\mathbf{h_1 h_5 - h_6})$, $(r = 2)$;
$2730h_1^6 + 3080h_1^4h_2 + 1064h_1^3h_3 + 35h_1^2h_4 - (\mathbf{h_1 h_5 - h_6})$, $(r = 3)$.

"almost" h-positivity (II)

Theorem (S, 2020)

For any nonempty rank set $T \subseteq [1,k]$*, consider the module*

$$\tilde{H}_{k-2}(A^*_{n,k}(T)) + (-1)^{|T|} S_{(n-1,1)}.$$

Its Frobenius characteristic $F_{n,k}(T) + (-1)^{|T|} s_{(n-1,1)}$ *is supported on the set* $T_2(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geq 2\}$ *with* **nonnegative** *integer coefficients in each of the following cases:*

1. $T = [r,k], k \geq r \geq 1.$
2. $T = [1,k] \backslash \{r\}, k \geq r \geq 1.$
3. $T = \{1 \leq s_1 < s_2 \leq k\}.$

Conjecture (B)

Let A be an alphabet of size $n \geq 2$. Then the homology of any finite nonempty rank-selected subposet of subword order on A^*, **plus or minus one copy of the reflection representation of** S_n, *is a permutation module. In fact the Frobenius characteristic is h-positive and supported on the set*

$$T_2(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geq 2\}.$$

A BIG THANK YOU TO
THE FPSAC2021 ORGANISERS,

AND

THANK YOU FOR COMING!

Theorem (S, 2020)

Fix $k \geq 1$. The kth tensor power of the reflection representation $S_{(n-1,1)}^{\otimes k}$, i.e. the homology module $\tilde{H}_{k-1}(A_{n,k}^*)$, has the following property: $S_{(n-1,1)}^{\otimes k} \oplus (-1)^k S_{(n-1,1)}$ is a permutation module $U_{n,k}$ whose Frobenius characteristic is h-positive, and is supported on the set $\{h_\lambda : \lambda = (n-r, 1^r), r \geq 2\}$. If $k = 1$, then $U_{n,1} = 0$. More precisely, the k-fold internal product $s_{(n-1,1)}^{*k}$ has the following expansion in the basis of homogeneous symmetric functions h_λ :

$$\sum_{d=0}^{n} g_n(k,d) h_1^d h_{n-d},$$

where $g_n(k,0) = (-1)^k, g_n(k,1) = (-1)^{k-1}$, and $g_n(k,d) = \sum_{i=d}^{k} (-1)^{k-i} S(i-1, d-1)$, for $2 \leq d \leq n$.

Tensor powers of $S_{(n-1,1)}$ – (II)

Hence $s_{(n-1,1)}^{*k} = (-1)^{k-1}(s(n-1,1) + \mathrm{ch}\,(U_{n,k})$, where $\mathrm{ch}\,(U_{n,k}) = \sum_{d=2}^{n} g_n(k,d) h_1^d h_{n-d}$.
The integers $g_n(k,d)$ are independent of n for $k \leq n$, nonnegative for $2 \leq d \leq k$, and $g_n(k,d) = 0$ if $d > k$. Also:

1. $g_n(k,2) = \frac{1+(-1)^k}{2}$.
2. $g_n(k,k-1) = \binom{k-1}{2} - 1$, $k \leq n$.
3. $g_n(k,k) = 1$, $k < n$.

Theorem (S, 2020)

The positive integer $\beta_n(k) = \sum_{d=2}^{\min(n,k)} g_n(k,d)$ *is the multiplicity of the trivial representation in* $S_{(n-1,1)}^{\otimes k}$*. When* $n \geq k$*, it equals the number of set partitions* $B_k^{\geq 2}$ *of the set* $\{1,\ldots,k\}$ *with no singleton blocks.*

This gives the stable dimension of the quotient complex.

Also $\beta_n(n+1) = B_{n+1}^{\geq 2} - 1$ *and* $\beta_n(n+2) = B_{n+2}^{\geq 2} - \binom{n+1}{2}$.

Enumerative consequences – (II)

Theorem (S, 2020; (?))

The first $n-1$ positive tensor powers of $S_{(n-1,1)}$ are an integral basis for the vector space spanned by the positive tensor powers. The nth tensor power of $S_{(n-1,1)}$ is an integer linear combination of the first $(n-1)$ tensor powers:

$$S_{(n-1,1)}^{\otimes n} = \bigoplus_{k=1}^{n-1} a_k(n) S_{(n-1,1)}^{\otimes k},$$

with $a_{n-1}(n) = \binom{n-1}{2}$.

Let $c(n, j)$ be the number of permutations in S_n with exactly j disjoint cycles.
A recurrence for the coefficients $a_k(n)$ is:

$$a_{n-1}(n) = \binom{n-1}{2};$$

$$(n-2)a_j(n) - a_{j-1}(n) = (-1)^{n-j}[c(n,j) - c(n,j-1)],$$

$$2 \le j \le n-1;$$

$$(n-2)a_1(n) = c(n,1)(-1)^{n-1}$$

$$\Longrightarrow a_1(n) = \frac{(n-1)!}{n-2}(-1)^{n-1} = (-1)^{n-1}[(n-2)! + (n-3)!]$$

Question

Recall that $a_{n-1}(n) = \binom{n-1}{2}$. Is there a combinatorial interpretation for the signed integers $a_i(n)$? There are many interpretations for $(-1)^{n-1}a_1(n) = (n-2)! + (n-3)!$, see OEIS A001048.

For $n \geq 4$ it is also the size of the largest conjugacy class in S_{n-1}. The other sequences $\{a_i(n)\}_{n\geq 3}$ are NOT in OEIS.

Some data

Example

Write X_n^k for $S_{(n-1,1)}^{\otimes k}$. Maple computations with Stembridge's SF package show that

1. $X_3^3 = X_3^2 + 2X_3$.
2. $X_4^4 = 3X_4^3 + X_4^2 - 3X_4$.
3. $X_5^5 = 6X_5^4 - 7X_5^3 - 6X_5^2 + 8X_5$.
4. $X_6^6 = 10X_6^5 - 30X_6^4 + 20X_6^3 + 31X_6^2 - 30X_6$
5. $X_7^7 = 15X_7^6 - 79X_7^5 + 165X_7^4 - 64X_7^3 - 180X_7^2 + 144X_7$
6. $X_8^8 = 21X_8^7 - 168X_8^6 + 630X_8^5 - 1029X_8^4 + 189X_8^3 + 1198X_8^2 - 840X_8$.

Question

For fixed k and n, what do the positive integers $g_n(k,d)$ count? Is there a combinatorial interpretation for $\beta_n(k) = \sum_{j=d}^{\min(n,k)} g_n(k,d)$, the multiplicity of the trivial representation in the top homology of $A^*_{n,k}$, in the nonstable case $k > n$?

Recall that for $k \leq n$, this is the number $B_k^{\geq 2}$ of set partitions of $[k]$ with no singleton blocks, and is sequence OEIS A000296.

Enumerative Questions – (III)

Proposition (S, 2020)

There are two formulas for $g_n(k,d)$:

$$\sum_{j=d}^{k}(-1)^{k-j}S(j-1,d-1)=\sum_{r=0}^{k-d}(-1)^{r}\binom{k}{k-r}S(k-r,d).$$

In particular, when $n \geq k$*, this multiplicity is independent of* n*.*

Question

Is there a combinatorial explanation?

Note: The blue formula shows that $g_n(k,d)$ is a nonnegative integer.

Techniques from [S, Adv. in Math 1994] and [S, Jerusalem Combinatorics, Contemp. Math, 1994]:

Theorem (S, 1994)

Equivariant acyclicity of Whitney homology.

Theorem (S, 1994)

A formula for finding the homology of subposets from the known homology of the poset P, e.g. by deleting an antichain.

Methods – II

Also, inspired by an observation of Richard Stanley:

Proposition (S, 2020)

Subword order belongs to a family of posets $\{P_n\}$ with automorphism group S_n such that the action of S_n is determined by the Möbius number $\mu(P_n)$ as a polynomial in n.

Hopf trace formula says that the trace of $g \in G$ on the Lefschetz module of a G-invariant poset P is the Möbius number of the fixed-point subposet P_g.