Horizontal-strip LLT polynomials

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Definition: Let $\lambda=(R_1,\dots,R_n)$ be a sequence of rows. The (horizontal-strip) LLT polynomial is

$$G_\lambda(x;q)=\sum_{T\in SSYT(\lambda)} q^{\mathrm{inv}(T)}x^T,$$

where $\mathrm{inv}(T)$ is the number of [illegible] or [illegible] (x,y).

Example: $\lambda=(4/0,5/4,8/5,6/1)$

$G_\lambda(x;q)=$ [illegible]

Theorem (Lascoux, Leclerc, Thibon '97): The LLT polynomial $G_\lambda(x;q)$ is a symmetric function.

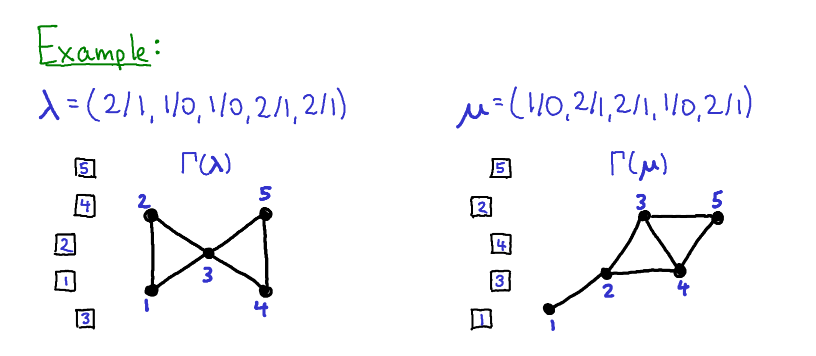
Theorem (Leclerc, Thibon '00), (Grojnowski, Haiman '07): The LLT polynomial $G_\lambda(x;q+1)$ is Schur-positive.

Problem 1: For which λ and μ do we have $G_\lambda(x;q)=G_\mu(x;q)$?

Problem 2: Find a combinatorial Schur expansion of $G_\lambda(x;q)$.

If λ is unicellular, we can associate a labelled unit interval graph $\Gamma(\lambda)$ to λ.

Example: $\lambda=$ [illegible] $\mu=$ [illegible]



Theorem (Carlsson, Mellit '18): If λ is unicellular, then $(q-1)^{-n}G_\lambda[x(q-1)]=X_{\Gamma(\lambda)}(x;q)$, the chromatic quasisymmetric function of $\Gamma(\lambda)$. In particular, if λ and μ are unicellular, then $G_\lambda(x;q)=G_\mu(x;q)$ if and only if $X_{\Gamma(\lambda)}(x;q)=X_{\Gamma(\mu)}(x;q)$.

Theorem (Huh, Nam, Yoo '20): If $\Gamma(\lambda)$ is a melting lollipop, then

$$G_\lambda(x;q)=\sum_{T} q^{[illegible]} s_{\mathrm{shape}(T)}.$$

Maximum number of inversions:

x can make an inversion with y or z but not both.

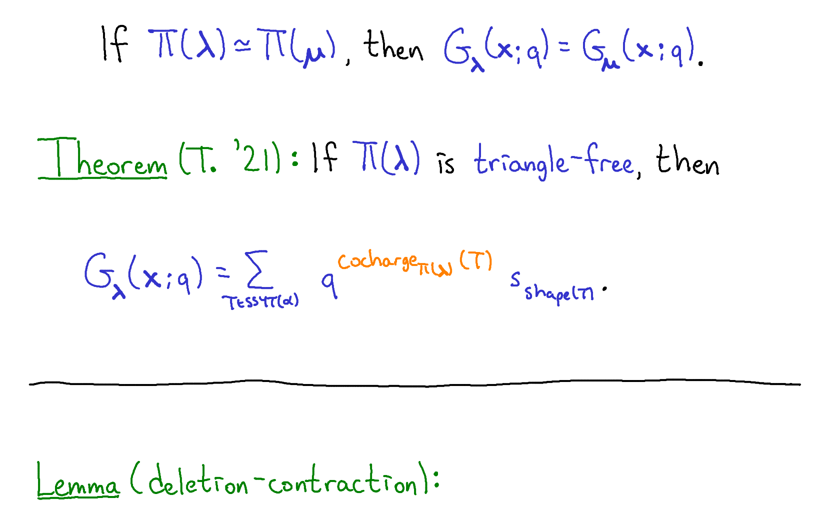
Definition (T. '21): Let R and R' be rows. We define

$$M(R,R')=\begin{cases}|R\cap R'| & \text{if } R \text{ starts weakly left of } R',\\ |R\cap R'^{+}| & \text{if } R \text{ starts strictly right of } R',\end{cases}$$

where R'^{+} is R' shifted right by one cell.

Let $\lambda=(R_1,\dots,R_n)$ be a horizontal-strip. We define a weighted graph $\Pi(\lambda)$.

Example: $\lambda=(4/0,5/4,8/5,6/1)$



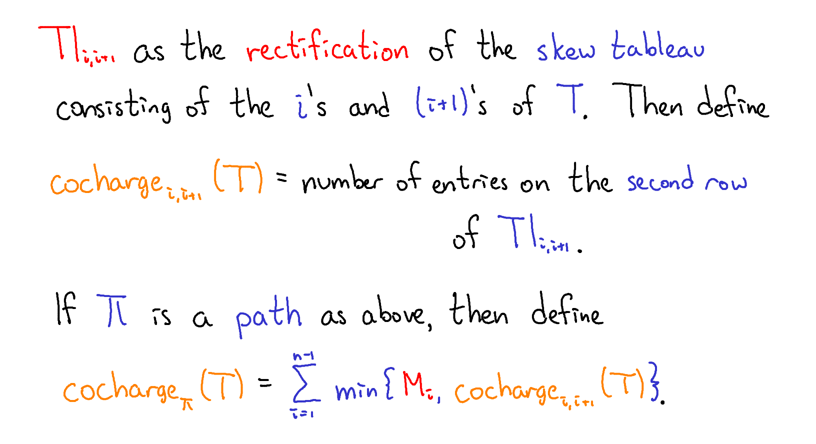
Theorem (T. '21+): Let λ and μ be horizontal-strips. If $\Pi(\lambda)\simeq\Pi(\mu)$, then $G_\lambda(x;q)=G_\mu(x;q)$.

Theorem (T. '21): If $\Pi(\lambda)$ is triangle-free, then

$$G_\lambda(x;q)=\sum_{T\in SSYT(\alpha)} q^{\mathrm{cocharge}_{\Pi(\lambda)}(T)} s_{\mathrm{shape}(T)}.$$

Lemma (deletion-contraction): If $R_i\leftrightarrow R_{i+1}$ and R_i starts strictly right of R_{i+1}, then

$$G_{(R_1,\dots,R_i,R_{i+1},\dots,R_n)}(x;q)=qG_{(R_1,\dots,R_{i+1},R_i,\dots,R_n)}(x;q)-(q-1)G_{(R_1,\dots,R_i\cup R_{i+1},R_i\cap R_{i+1},\dots,R_n)}(x;q)$$



Paths: $(\alpha_1)\overset{M_1}{—}(\alpha_2)\overset{M_2}{—}(\cdots)—(\cdots)—(\cdots)\overset{M_{n-1}}{—}(\alpha_n)$

Definition: Let $T\in SSYT(\alpha)$ and $i\geq 1$. We define $T|_{i,i+1}$ as the rectification of the skew tableau consisting of the i's and $(i+1)$'s of T. Then define

$\mathrm{cocharge}_{i,i+1}(T)$ = number of entries on the second row of $T|_{i,i+1}$.

If Π is a path as above, then define

$$\mathrm{cocharge}_\pi(T)=\sum_{i=1}^{n-1}\min\{M_i,\mathrm{cocharge}_{i,i+1}(T)\}.$$

Example: $\lambda=(11/7,7/5,6/3,4/0)$

$\Pi(\lambda)=(4)\overset{2}{—}(3)\overset{2}{—}(2)\overset{1}{—}(4)$

The coefficient of s_{733} is calculated as follows.

$S=$

4	4	4				
2	3	3				
1	1	1	1	2	2	4

$S|_{12}\to 1$, $S|_{23}\to 2$, $S|_{34}\to \min\{1,2\}=1$; $\mathrm{cocharge}_\pi=4$

$T=$

4	4	4				
2	2	3				
1	1	1	1	2	3	4

$T|_{12}\to 2$, $T|_{23}\to 1$, $T|_{34}\to\min\{1,2\}=1$; $\mathrm{cocharge}_\pi=4$

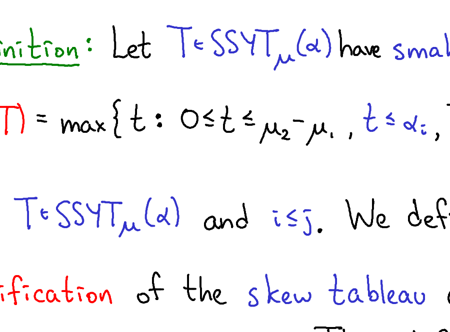
$U=$

4	4	4				
2	2	2				
1	1	1	1	3	3	4

$U|_{12}\to 2$, $U|_{23}\to 0$, $U|_{34}\to\min\{1,2\}=1$; $\mathrm{cocharge}_\pi=3$

$(2q^4+q^3)s_{733}$

More generally, for triangle-free graphs:



$T=$

2	2	2	2	2						
1	1	1	1	1	1	1	2	2	2	

$\mathrm{cocharge}_{1,2}(T)$ = number of entries on the second row $=5$
= number of 1's − distance we can move left
$=7-2=5$

Definition: Let $T\in SSYT_\mu(\alpha)$ have smallest entry i. We define

$$f(T)=\max\{t: 0\leq t\leq \mu_2-\mu_1,\ t\leq\alpha_i,\ T_{2,j}>T_{1,j'+t}\ \forall\, 1\leq j'\leq\mu_2\}.$$

Let $T\in SSYT_\mu(\alpha)$ and $i\leq j$. We define $T|_{i,j}$ as the rectification of the skew tableau consisting of the entries x with $i\leq x\leq j$. Then define

$$\mathrm{cocharge}_{i,j}(T)=\alpha_i-f(T|_{i,j})\quad\text{and}$$

$$\mathrm{cocharge}_\pi(T)=\sum_{i<j}\min\{M_{i,j},\mathrm{cocharge}_{i,j}(T)\}.$$

Example: $\lambda=(4/0,5/4,8/5,6/1)$

$\Pi(\lambda)=$ $v_1(4)\overset{3}{—}v_3(5)\overset{2}{—}v_4(3)$, $v_3(5)\overset{1}{—}v_2(1)$

The coefficient of s_{733} is calculated as follows.

$S=$

4	4	4				
3	3	3				
1	1	1	1	2	3	3

$S|_{13}$: $4-2=2$; $S|_{23}$: 1; $S|_{34}$: $\min\{2,3\}=2$ → 5

$T=$

4	4	4				
2	3	3				
1	1	1	1	3	3	3

$T|_{13}$: $4-1=3$; $T|_{23}$: 0; $T|_{34}$: 2 → 5

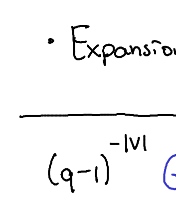
$U=$

3	4	4				
2	3	3				
1	1	1	1	3	3	4

$U|_{13}$: $4-1=3$; $U|_{23}$: 1; $U|_{34}$: 2 → 6

$(q^6+2q^5)s_{733}$

Thank you!



Further directions:

- Generalize $\mathrm{cocharge}_\pi$ to more graphs π
- Define an extended chromatic quasisymmetric function associated to a weighted labelled graph
- k-Schur function expansions
- Expansions of Macdonald polynomials

$$(q-1)^{-|V|}G_\lambda[x(q-1)]\Big|_{q=1}=X_{\pi(\lambda)}(x)$$ – extended chromatic symmetric function (Crew, Spirkl '20)

α – composition; P_α – path with vertex weights α, edge weights 1

P_{21231}: (2)—(1)—(2)—(3)—(1)

P_{23121}: (2)—(3)—(1)—(2)—(1)

α,β – compositions. TFAE

- $G_{P_\alpha}=G_{P_\beta}$
- $X_{P_\alpha}=X_{P_\beta}$
- $r_\alpha=r_\beta$
- $\mathcal{M}(\alpha)=\mathcal{M}(\beta)$

$\Downarrow$

$G_{\alpha\circ\pi}=G_{\beta\circ\pi}$