The Generating Function For Interacting Kreweras Walks Is Not Algebraic

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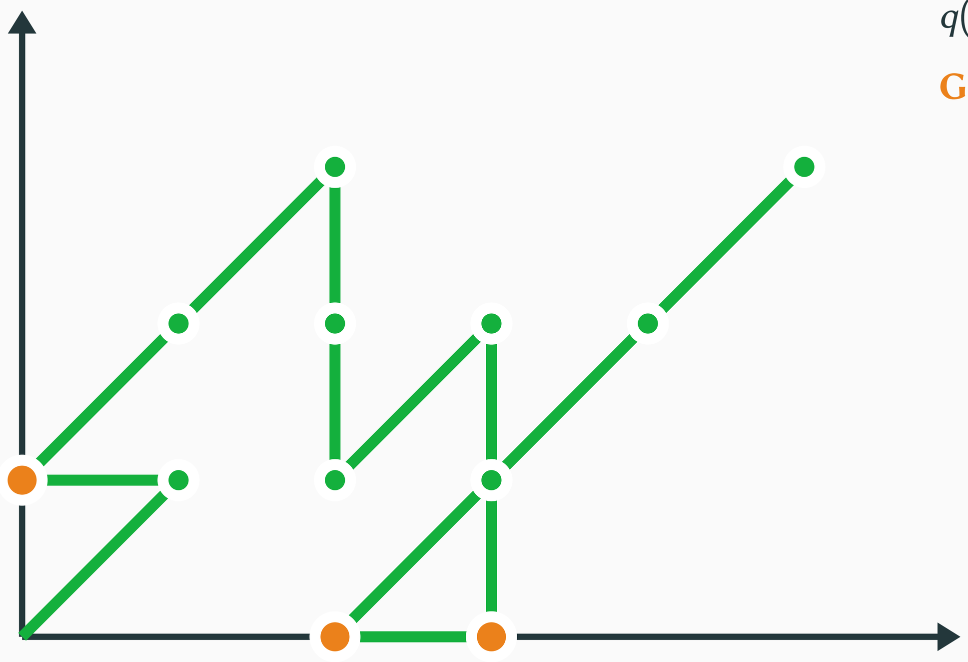

$q(n; i, j; h, v, u)$ = # paths of length n ending in (i, j) and touching $h/v/u$ times the x axis / the y axis / the origin

Generating series $Q(a, b, c; x, y; t) = \sum_{h,v,z,i,j,n} q(n; i, j; h, v, u) a^h b^v c^u x^i y^j t^n$

- Knowing that a generating series is D-finite (annihilated by a differential equation) or algebraic (annihilated by a polynomial) gives access to useful information, and allows to use more efficient algorithms.
- In [1], Beaton et al. studied Kreweras and reverse Kreweras walks counting how many times they hit the boundaries of the domain. The combinatorics of such walks have applications in statistical physics [5].
- They proved that the generating series for reverse Kreweras walks is algebraic, and the generating series for Kreweras walks is D-finite.
- They also conjectured that the generating series for Kreweras walks is not algebraic.
- We prove this conjecture using tools from computer algebra.

$Q(a, b, c; x, y; t)$ → $\Theta(x; t) = [x^> y^0] \left(\frac{(x-y)(x^2y-1)(xy^2-1)}{xy(1-t(xy+x^{-1}+y^{-1}))} \right) =: [x^> y^0] \Theta_0(x, y; t)$

The proof that L exists [1] is constructive, but the equation is large. Even if it could be computed, it would not be suitable for the rest of the proof.

Coefficient and positive part extractions can be expressed in terms of residues:

$$\Theta = \mathrm{Res}_{z=0} \mathrm{Res}_{y=0} \left[\frac{1}{yz} \Theta_0(z, y; t) \frac{\frac{x}{z}}{1 - \frac{x}{z}} \right]$$

An annihilating differential equation can be computed using **Creative Telescoping** [3]. The result is correct but not minimal.

An annihilating differential equation for a power series approximation of Θ can be computed by **Guessing** [2]. This allows to find an equation with minimal order.

Differential equation L with $L(Q) = 0$

Differential equation L_1 with $L_1(\Theta) = 0$

Differential equation L_2 with $L_2(\Theta) = O(T^N)$

Using **closure properties** and **operator factorization**, we reduce the problem to solving simpler equations. The last step is solving a differential equation of order 3 with hypergeometric solutions, using a **dedicated solver** [4].

Closed form C with $L(C) = 0$

Closed form C with $L_1(C) = 0$

Closed form C with $L_2(C) = 0$

Exact verification: evaluate $L_1(C)$ formally or check that L_2 is a right divisor of L_1.

Symbolic-numeric verification: C and Θ annihilate the same differential equation L_1, so it is enough to prove that their power series approximations agree.

Using **closure properties**, we only need to decide whether the hypergeometric functions are algebraic. It can be done by looking up the functions in Schwartz's classification, or by **analyzing the singularities** of the differential operator.

$C = Q$

C is not algebraic

Theorem: $\Theta = C$

C is not algebraic

Theorem: Q is not algebraic

Θ is not algebraic

Further questions:

- Those results apply if $a \neq b$ and $c \neq 0$ (and in particular if they are symbolic variables). The case $a = b$ is known [1]. On the other hand, if $c = 0$, the series is still D-finite, but we do not know if it is algebraic.
- The generating series $Q(a, b, c; 1, 1; t)$ counting walks regardless of their ending point is also interesting. Experimentally, this series appears to be algebraic, and annihilated by a polynomial of degree 24 in Q, 92 in t, 60 in a and b and 24 in c.

References:

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