MULTICRITICAL RANDOM PARTITIONS

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Universal edge fluctuations

The **Tracy–Widom** distribution characterizes the interfaces of a diverse class of statistical models. Discovered as the asymptotic distribution of the largest eigenvalue $\lambda_{\max}$ of a Gaussian random $N \times N$ Hermitian matrix $\mathbb{P}(M) \propto e^{-\frac{1}{2}\operatorname{tr} M^2}$,

$$F_{TW}(s) = \lim_{N\to\infty} \mathbb{P}\left(\frac{\lambda_{\max} - \sqrt{2N}}{\sqrt{2}N^{-1/6}} \le s\right)$$

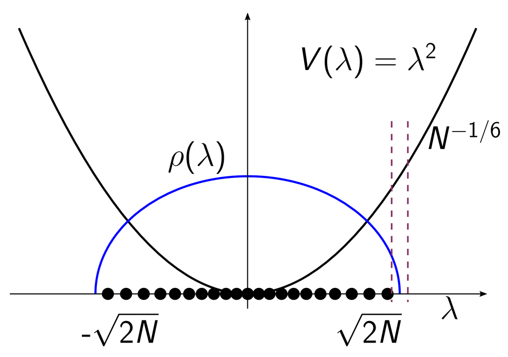

Fig. 1: Eigenvalues of Gaussian random Hermitian matrices

governs, for instance,

- height fluctuations in random growth models (Prähofer & Spohn, 2000)
- domino statistics in the Aztec diamond (Johansson, 2005)
- the length of the longest increasing subsequence of a random permutation (Baik, Deift & Johansson, 1999).

...and equivalently, via the RSK bijection, the first part of a Plancherel random partition

Such models typically map to **free fermions in one dimension** (or **DPPs**) where all correlations are determinants of a kernel. The TW distribution is a τ-function of the **Painlevé II** equation. Alternative edge statistics with the same integrable structure have been introduced using Hermitian matrix models (Claeys, Its & Krasovsky, 2010), Lax operators (Akemann & Atkin, 2012) and recently fermion momenta (Le Doussal, Majumdar & Schehr, 2018). This last approach found τ-functions of higher-order Painlevé II hierarchy equations, coinciding with previously studied multicritical unitary matrix integrals (Periwal & Shevitz, 1990).

By studying Schur random partitions with the same edge statistics, we explain the origin of this connection. These models map to many other ones, helping to explore the new universality classes.

Multicritical edges in fermion momenta

Fermion momenta in **flat-traps** $V(x) = x^{2n}$ follow higher-order analogues of the TW distribution near the Fermi edge (LDMS 2018). The edge Hamiltonian

$$H = \int_{\mathbb{R}} dp\ \psi^*(p)\left[p + (-1)^n \frac{d^{2n}}{dp^{2n}}\right]\psi(p)$$

has modes $\chi(\mu) = \int_{\mathbb{R}} dp\ \mathrm{Ai}_{2n+1}(p+\mu)\psi(p)$ where

$$\mathrm{Ai}_{2n+1}(p) = \int_{i\mathbb{R}+\delta} \frac{d\zeta}{2\pi i} e^{\frac{(-1)^{n+1}}{2n+1}\zeta^{2n+1} - p\zeta}$$

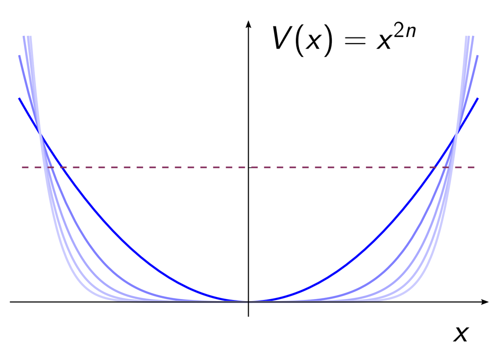

Fig. 2: Flat-trap potentials

are the generalized Airy functions. The kernel is

$$\mathcal{A}_{2n+1}(p,p') = \langle \psi(p)\psi^*(p')\rangle = \int_0^\infty d\mu\ \mathrm{Ai}_{2n+1}(p+\mu)\,\mathrm{Ai}_{2n+1}(p'+\mu)$$

and the law of the largest momentum is the **Fredholm determinant**

$$\mathbb{P}(p_{\max} < s) = \lim_{\alpha\to-\infty} \langle e^{\alpha\int_s^\infty dp\,\psi(p)\psi^*(p')}\rangle = \det(1 - \mathcal{A}_{2n+1})_{[s,\infty)} := F(2n+1; s),$$

with $\det(1-\mathcal{A}_{2n+1})_{[s,\infty)} = 1 + \sum_{n=0}^\infty \frac{(-1)^n}{n!}\int_s^\infty dx_1 \ldots \int_s^\infty dx_n\ \det_{1\le i,j\le n}\mathcal{A}_{2n+1}(x_i,x_j)$.

At $n=1$ this is TW distribution; indeed, GUE eigenvalues correspond to fermions in a harmonic potential

Integer partitions and Schur measures

Integer partitions $\lambda = (\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n > 0)$ map to 1D fermionic collective states $S(\lambda) = \{\lambda_i - i + \frac{1}{2} | i \in \mathbb{Z}_{>0}\}$; **Schur measures** (Okounkov, 2000) define DPPs on them. Constructing states from $\Gamma_\pm(t) = e^{\sum_r \frac{t_r}{r} a_{\pm r}}$, $a_r = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_{k-r}\psi_k^*$, we consider correlations

$$\rho(X) = e^{-\sum_r \frac{t_r t_r'}{r}} \langle\emptyset|\Gamma_+(t)\prod_{k\in X}\psi_k\psi_k^*\Gamma_-(t')|\emptyset\rangle = \det_{k,\ell\in X} K(k,\ell), \qquad X \subset \mathbb{Z}_{>0} + \tfrac{1}{2}$$

where the kernel is generated by

$$\sum_{k,l\in\mathbb{Z}+\frac{1}{2}} z^k w^{-\ell} K(k,\ell) = \mathbf{K}(z,w) = \frac{e^{\sum_{r>0}\frac{1}{r}(t_r z^r - t_r' z^{-r})}}{e^{\sum_{r>0}\frac{1}{r}(t_r w^r - t_r' w^{-r})}}\frac{\sqrt{zw}}{z-w}, \qquad \text{for } |w| < |z|.$$

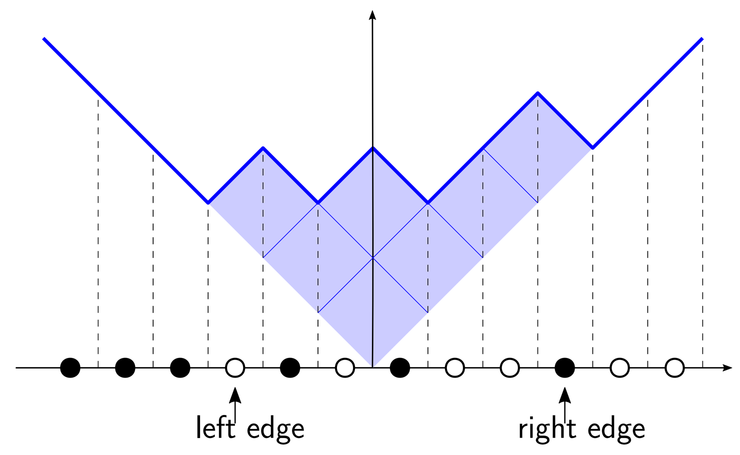

Fig. 3: Young diagram of $\lambda = (4,2,1)$ in the Russian convention, and the corresponding Maya diagram for $S(\lambda)$

Where $s_\lambda[t_1, t_2, \ldots]$ are the Schur polynomials in the power sum variables $\{t_r\}$,

$$\mathbb{P}(\lambda) = \rho(\{\lambda_i - i + \tfrac{1}{2} | i \in \mathbb{Z}_{>0}\}) = e^{-\sum_r \frac{t_r t_r'}{r}} s_\lambda[t_1, t_2, \ldots] s_\lambda[t_1', t_2', \ldots].$$

Multicritical Schur measures

We introduce multicritical measures

$$\mathbb{P}^\gamma_{n,\theta}(\lambda) = e^{-\theta^2\sum_r \frac{\gamma_r^2}{r}} s_\lambda[\theta\gamma_1, \theta\gamma_2, \ldots] s_\lambda[\theta\gamma_1, \theta\gamma_2, \ldots]$$

on integer partitions, where $\{\gamma_r\}$, along with the edge and scaling parameters b, d, satisfy

$$2\sum_r r^{2p}\gamma_r = \delta_{p,0} b + \delta_{p,n}(-1)^{n+1}(2n)!\,d\,, \qquad p = 0,1,2,\ldots,n.$$

Theorem *Let λ be a random partition distributed by $\mathbb{P}^\gamma_{n,\theta}$. Then the asymptotic distribution of its first part is*

$$\lim_{\theta\to\infty} \mathbb{P}^\gamma_{n,\theta}\left[\frac{\lambda_1 - b\theta}{(d\theta)^{\frac{1}{2n+1}}} < s\right] = F(2n+1; s) = \det(1 - \mathcal{A}_{2n+1})_{[s,\infty)}.$$

The $n=1$ case, $F(3,s) = F_{TW}(s)$, is the most generic, with $\{\gamma_r\}$ unconstrained. This generalizes the **Baik–Deift–Johansson** theorem. We study two families of minimal multicritical measures:

- **"Odd-even" measure** $\mathbb{P}^{\mathrm{oe}}_{n,\theta}(\lambda)$ defined as above, for $\gamma_r = (-1)^r\binom{2n}{n-r}/\binom{2n}{n-1}$, $r = 1,\ldots,n$, and all other $\gamma_r = 0$, $b = \frac{n+1}{n}$ and $d = \binom{2n}{2n+1}$
- **"Odd" measure** $\mathbb{P}^{\mathrm{o}}_{n,\theta}(\lambda)$ defined as above, for $\gamma_{2r-1} = (-1)^r\binom{2n-1}{n-r}/(\binom{2n-1}{n-1}(2r-1))$, $r = 1,\ldots,n$, and all other $\gamma_r = 0$, $b = 2^{4n-1}/n\binom{2n}{n}^2$ and $d = \frac{(2n-1)!!}{(2n-2)!!}$

Lemma *Let λ be distributed by a Schur measure $\mathbb{P}^\gamma_{n,\theta}$. Then its conjugate partition is distributed by $\mathbb{P}^{\tilde\gamma}_\theta$, where $\tilde\gamma_r = (-1)^{r-1}\gamma_r$.*

For λ distributed by $\mathbb{P}^{\mathrm{o}}_{n,\theta}$, it follows that its length $\ell(\lambda)$ has the same law as λ_1; under $\mathbb{P}^{\mathrm{oe}}_{n,\theta}(\lambda)$, the asymptotic law of $\ell(\lambda)$ is $F(3,s) = F_{TW}(s)$ for all n.

Limit shapes of large Young diagrams

As $\theta \to \infty$, the Russian Young diagram profile of a random partition concentrates to a limit shape $\Omega(x)$, related to the limiting density profile $\rho^\gamma_{n,\theta}(x) = \lim_{\theta\to\infty}\sum_{\lambda:x\theta\in S(\lambda)}\mathbb{P}^\gamma_{n,\theta}(\lambda) = \lim_{\theta\to\infty} K(x\theta, x\theta)$ by $\Omega'(x) = 1 - 2\rho(x)$. For the odd-even and odd measures,

$$\rho^{\mathrm{oe}}(x) = \tfrac{1}{\pi}\arccos\left(1 - \tfrac{1}{2}\binom{2n}{n-1}^{\frac{1}{n}}\left(\tfrac{n+1}{n} - x\right)^{\frac{1}{n}}\right), \qquad x \in \left[-\tfrac{n+1}{n}\left(\tfrac{(2n)!!}{(2n-1)!!} - 1\right), \tfrac{n+1}{n}\right]$$

$$\rho^{\mathrm{o}}(x) = \frac{\chi(x)}{\pi}, \quad \int_0^{\chi(x)} dx\,(2\sin\phi)^{2n-1}\phi = (-1)^{n+1}\binom{2n-1}{n}x, \qquad x \in \left[-\frac{2^{4n-1}}{n\binom{2n}{n}^2}, \frac{2^{4n-1}}{n\binom{2n}{n}^2}\right].$$

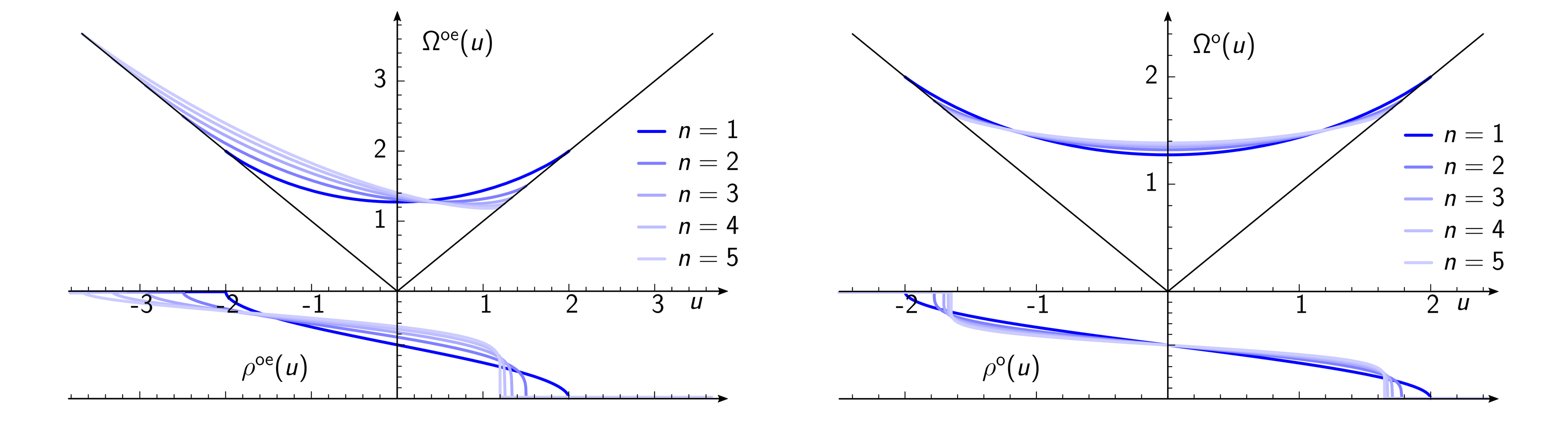

Fig. 4: Limit shape and density profile of $\mathbb{P}^{\mathrm{oe}}_{n,\theta}$ (left) and $\mathbb{P}^{\mathrm{o}}_{n,\theta}$ (right) distributed random partitions, with $\frac{1}{2n}$ vanishing. Note the symmetry of $\rho^{\mathrm{o}}(x)$.

Unitary matrix integrals via Toeplitz determinants

Proposition *For λ distributed by $\mathbb{P}^\gamma_{n,\theta}$, the law of its first part satisfies*

$$e^{\theta^2\sum_r\frac{\gamma_r^2}{r}}\mathbb{P}^\gamma_{n,\theta}[\lambda_1 < \ell] = \det_{1\le i,j\le\ell}[g_{j-i}] = \int_{\mathcal{U}(\ell)} dU\ \exp\left[\theta\operatorname{tr}\sum_r \frac{(-1)^{r-1}\gamma_r}{r}(U^r + U^{*r})\right]$$

where the middle Toeplitz determinant has symbol $\sum_{k\in\mathbb{Z}} g_k z^k = \exp\sum_r \theta\frac{(-1)^{r-1}\gamma_r}{r}(z^r + z^{-r})$.

This follows from identities for Toeplitz determinants (Gessel, 1990; Heine). The unitary matrix integrals found for $\mathbb{P}^{\mathrm{oe}}_{n,\theta}$ have been related to the Painlevé II hierarchy (PS, 1990), just as the higher-order TW distributions $F(2n+1,s)$ have (LDMS, 2018; Cafasso, Claeys & Girotti, 2019).

In the $n=1$ case this is the Gross–Witten–Wadia model; in general, it indicates that $F(2n+1;s)$ corresponds to an order $2+\frac{1}{n}$ phase transition

Asymptotic analysis of the kernel

The asymptotic laws for $\mathbb{P}^\gamma_{n,\theta}$ are found from $K(k_i^\theta, k_j^\theta)$ at $k_i^\theta = b\theta + x_i(d\theta)^{1/(2n+1)}$;

$$K(k_i^\theta, k_j^\theta) = \oint\!\!\oint dzdw\,\mathbf{K}(z,w)e^{-k_i^\theta\log z + k_j^\theta\log w}.$$

We perform the double contour integral along $z = e^{\zeta\theta^{-1/(2n+1)}}$, $\zeta \in i\mathbb{R}+\delta$ (resp. w), and it is dominated by an order $2n$ saddle point at $z=1$, $w=1$; studying $\mathbf{K}(z,w)$ near this point we find $\lim_{\theta\to\infty} K(k_i^\theta, k_j^\theta) = \mathcal{A}_{2n+1}(x_i,x_j)$.

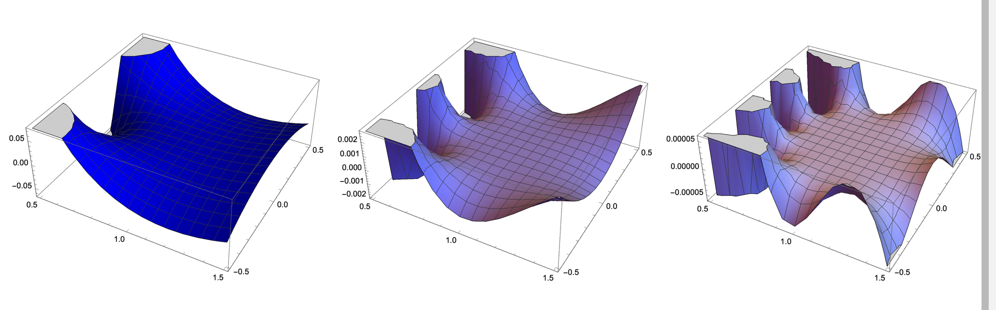
Fig. 5: $\mathrm{Re}(\sum_r \frac{\gamma_r}{r}(z^r - z^{-r}) - b\log z)$ near $z=1$ for $n = 1,2,3$

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