The Hurwitz action in complex reflection groups

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Introduction

- A complex reflection group is a finite group acting on a finite-dimensional complex vector space that is generated by complex reflections: non-trivial elements that fix a complex hyperplane pointwise.
- A particular family of complex reflection groups is $G(m,p,n)$, whose elements are the $n \times n$ monomial matrices: only one non-zero entry, a m-th root of unity, in each row and each column; and the product of these non-zero entries is a $\frac{m}{p}$-th root of unity.
- In particular, $G(m,1,n) \cong S_n \wr (\mathbb{Z}/m\mathbb{Z})$,where $\wr$ means the wreath product.

Hurwitz action

Let G be a group. Consider the following move:

$$\begin{array}{rl} \sigma_i : & (r_1, \ldots, \quad r_i, \quad r_{i+1}, \ldots, r_n) \in G^n \\ \mapsto & (r_1, \ldots, \quad r_{i+1}, \quad r_{i+1}^{-1} r_i r_{i+1}, \ldots, r_n) \end{array}$$

- Hurwitz action preserves product $g := r_1 \cdots r_n$, so it is an action on *factorizations* of g.
- Baumeister-Gobet-Roberts-Wegener [1] characterized elements in a finite real reflection group whose minimum-length factorizations form a single orbit under the Hurwitz action.

Cycle partitions

For $g \in G(m,p,n)$, a *cycle partition* Π of g is a set partition of the set of cycles of g such that for every part in Π, the corresponding cycle weights sum to $0 \pmod{p}$.

Let $|\Pi|$ denote the number of parts in Π. The *value* $v(\Pi)$ is the sum of $|\Pi|$ and the number of parts in Π of weight $0 \pmod{m}$. Π is *maximum* if its value is the largest possible. Let $\mathrm{Par}_{\max}(g)$ denote the set of maximum cycle partitions of g.

Theorem (Shi [2]). Let $g \in G(m,p,n)$. Its reflection length is $\ell(g) = n + \mathrm{cyc}(g) - v(\Pi)$, where $\mathrm{cyc}(g)$ is the number of cycles of g and $\Pi \in \mathrm{Par}_{\max}(g)$.

Results

Definition 1. Let $g \in G(m,p,n)$, let Π be a cycle partition of g, and let B be a part in Π. Suppose that the weights of the cycles in B are $(k_1, k_2, \ldots, k_{|B|})$. Define $r(B) = \gcd(m, k_1, k_2, \ldots, k_{|B|-1}, k_{|B|})$.

Theorem 2. *Given an element $g \in G(m,p,n)$, the number of Hurwitz orbits of its minimum-length factorizations is given by*

$$\sum_{\Pi \in \mathrm{Par}_{\max}(g)} \prod_{B \in \Pi} (r(B))^{|B|-1}.$$

Corollary 3. *Let $g \in G(m,p,n)$. The minimum-length factorizations of g form a single orbit under the Hurwitz action if and only if (1) $|\mathrm{Par}_{\max}(g)| = 1$, and (2) either $|B| = 1$ or $r(B) = 1$ for every part B in $\Pi \in \mathrm{Par}_{\max}(g)$.*

In particular, if g has a single cycle, then g is Hurwitz transitive (i.e., a single Hurwitz orbit). Further, when $p = 1$, g is Hurwitz transitive if and only if no two cycles of g have nonzero weights that sum to $0 \pmod{m}$.

Theorem 4. *Let G be any finite complex reflection group with the set of reflections T. Let $g \in G$ such that $\ell_T(g) = \mathrm{rank}(G)$. If g has a minimum-length factorization whose factors generate G, then all minimum-length factorizations of g generate G.*

Standard form factorizations

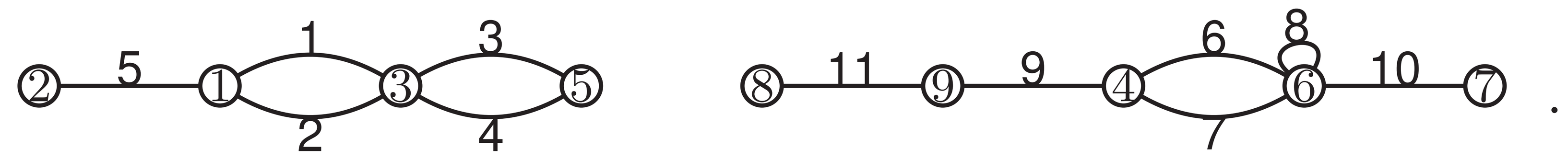
- Why standard form factorizations?

It is a key tool used in the proof of Theorem 2.

Example. The factorization $[(13);0] \cdot [(13);1] \cdot [(35);1] \cdot [(35);3] \cdot [(12);0] \cdot [(46);0] \cdot [(46);4] \cdot [\mathrm{id};(0,0,0,0,0,3,0,0,0)] \cdot [(49);1] \cdot [(67);2] \cdot [(89);3]$ of the element

$$[(12)(3)(498)(5)(67);(0,1,1,1,7,2,6,6,6)] \in G(9,3,9)$$

where $\Pi = [(12)(3)(5) \mid (498)(67)]$ is in standard form, with factorization graph



- What is special about standard form factorizations?

Lemma. For any $g \in G(m,p,n)$ and any minimum-length factorization f of g, there is a standard form factorization of g that is Hurwitz-equivalent (i.e., in the same Hurwitz orbit) to f.

In addition, given $g \in G(m,p,n)$ and a cycle partition Π, one can build a standard form factorization associated with Π.

- How are standard form factorizations related to Hurwitz orbits?

Lemma. For any $g \in G(m,p,n)$ and any two minimum-length standard form factorizations. They are in the same Hurwitz orbit if and only if they satisfy certain number theoretic condition.

Thus, given a maximum Π, to count the Hurwitz orbits, it suffices to count the standard form factorizations satisfying the number theoretic condition.

Example

In $G(4,2,3)$,

let $g = [(1)(2)(3)(4);(1,2,2,3)] = \begin{bmatrix} \zeta & & & \\ & \zeta^2 & & \\ & & \zeta^2 & \\ & & & \zeta^3 \end{bmatrix}.$

It has the following cycle partitions and values.

Π	$v(\Pi)$
$(2) \mid (3) \mid (1)(4)$	3+1
$(2)(3) \mid (1)(4)$	2+2
$(2) \mid (1)(3)(4)$	2+0
$(3) \mid (1)(2)(4)$	2+0
$(1)(2)(3)(4)$	1+1

The maximum value $v(\Pi)$ is 4 and its reflection length is $\ell(g) = 4 + 4 - 4 = 4$.

Observe that there are two cycle partitions having the maximum value, then the number of Hurwitz orbits is

$$(\gcd(4,2))^{1-1} \cdot (\gcd(4,2))^{1-1} \cdot (\gcd(4,1,3))^{2-1} + (\gcd(4,2,2))^{2-1} \cdot (\gcd(4,1,3))^{2-1} = 1 + 2 = 3.$$

For each orbit, we give a representative standard form factorization.

- $\Pi = [(2) \mid (3) \mid (1)(4)]$

$$\begin{bmatrix} 1 & & & \\ & \zeta^2 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \zeta^2 & \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{bmatrix} \cdot \begin{bmatrix} & & & \zeta^3 \\ & 1 & & \\ & & 1 & \\ \zeta & & & \end{bmatrix}$$

- $\Pi = [(2)(3) \mid (1)(4)]$

$$\begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & & \zeta^2 & \\ & \zeta^2 & & \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{bmatrix} \cdot \begin{bmatrix} & & & \zeta^3 \\ & 1 & & \\ & & 1 & \\ \zeta & & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ & & \zeta^3 & \\ & \zeta & & \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & & \zeta & \\ & \zeta^3 & & \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{bmatrix} \cdot \begin{bmatrix} & & & \zeta^3 \\ & 1 & & \\ & & 1 & \\ \zeta & & & \end{bmatrix}$$

References

[1] Barbara Baumeister, Thomas Gobet, Kieran Roberts, and Patrick Wegener, *On the Hurwitz action in finite Coxeter groups*, J. Group Theory **20** (2017), no. 1, 103–131.

[2] Jian-yi Shi, *Formula for the reflection length of elements in the group $G(m,p,n)$*, Journal of Algebra **316** (2007), no. 1, 284–296.