Gröbner Geometry of Schubert Polynomials Through Ice

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Joint work with
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Schubert Polynomials

The complete flag variety $GL(n)/B$ has a special family of subvarieties $\{X_w : w \in S_n\}$ called Schubert varieties.

Each Schubert variety defines a Schubert class $\sigma_w \in H^*(GL(n)/B)$. The Schubert classes are a linear basis for this ring.

$$\sigma_u \cdot \sigma_v = \sum_w c_{uv}^w \sigma_w$$

Littlewood - Richardson coefficients

By Borel's isomorphism:

$$H^*(GL(n)/B) \cong \mathbb{Z}[x_1, x_2, \dots, x_n]/I^{S_n}.$$

Lascoux and Schützenberger (1982) defined Schubert polynomials $\{\mathfrak{S}_w(x) : w \in S_n\}$ which are a choice of coset representatives for the images of the Schubert classes under Borel's isomorphism.

There's also double Schubert polynomials $\{\mathfrak{S}_w(x;y) : w \in S_n\}$ which are enriched versions of single Schuberts and satisfy

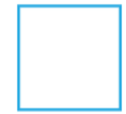
$$\mathfrak{S}_w(x) = \mathfrak{S}_w(x;0).$$

They tell you about T-equi cohomology

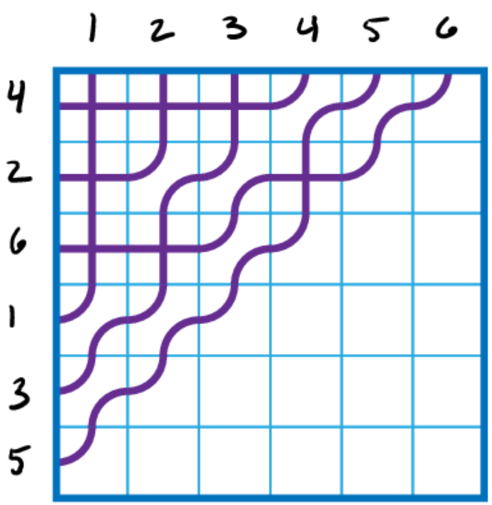
Pipe Dreams

on main antidiagonal

above main antidiagonal



below main antidiagonal



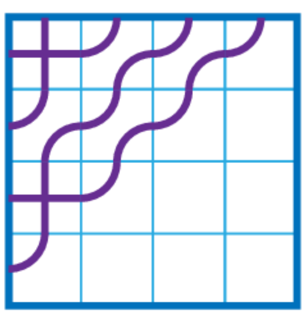
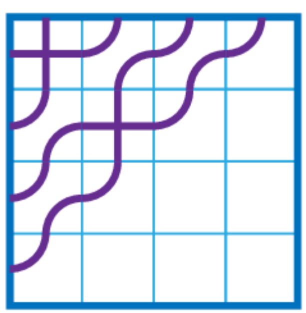
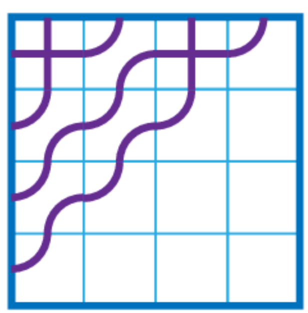
Fill $n \times n$ grid with n pipes that start at the top and end at the left and pairwise cross at most one time.

$$Wt(P) = (x_1-y_1)(x_1-y_2)(x_1-y_3)(x_2-y_1)(x_2-y_4)(x_3-y_1)(x_3-y_2)$$

Theorem (Fomin-Kirillov 1996 / Bergeron-Billey 1993):

The double Schubert polynomial is

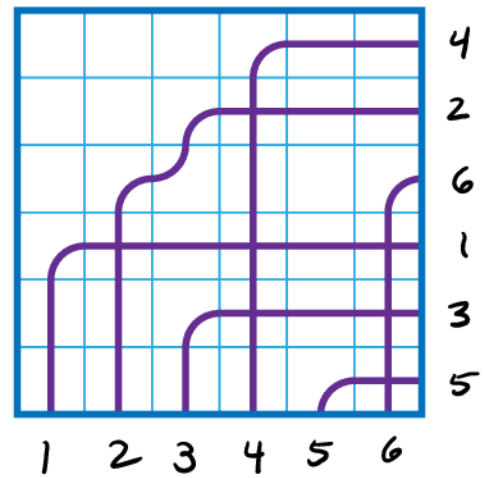
$$\mathfrak{S}_w(x;y) = \sum_{P \in \mathrm{Pipes}(w)} \mathrm{wt}(P).$$

$$\mathfrak{S}_{2143}(x;y) = (x_1-y_1)(x_3-y_1) \quad + \quad (x_1-y_1)(x_2-y_2) \quad + \quad (x_1-y_1)(x_1-y_3)$$

Bumpless Pipe Dreams

Fill $n \times n$ grid with n pipes that start at the bottom, end at the right and pairwise cross at most one time.



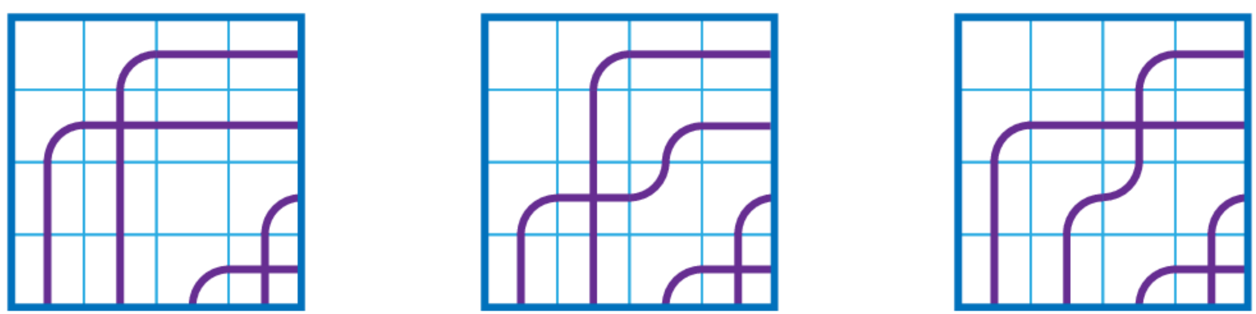
no bumps allowed!

$$wt(P) = (x_1 - y_1)(x_1 - y_2)(x_1 - y_3) \cdot (x_2 - y_1)(x_2 - y_2)(x_3 - y_1)(x_3 - y_5)$$

Theorem (Lam–Lee–Shimozono 2018):

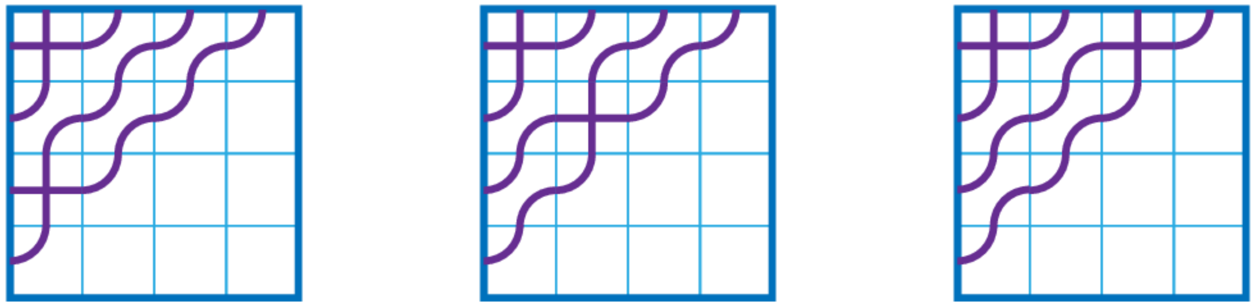
The double Schubert polynomial is

$$\mathfrak{S}_w(x;y) = \sum_{P \in BPD(w)} wt(P).$$

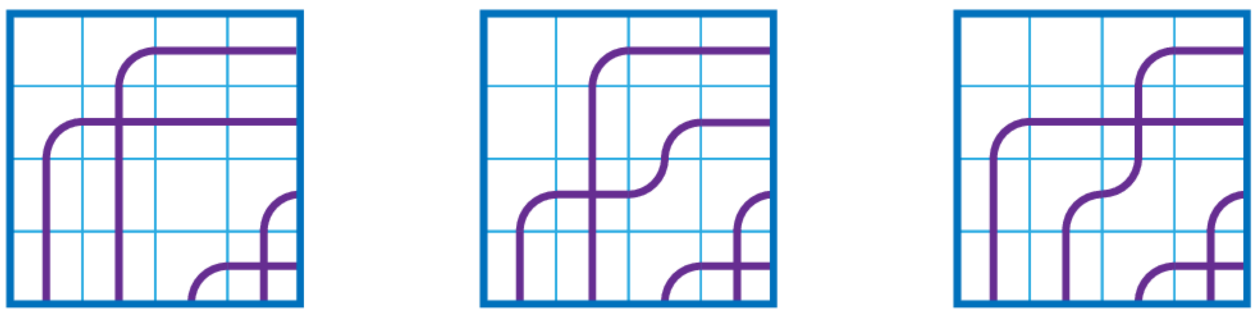


$$\mathfrak{S}_{2143}(x;y) = (x_1-y_1)(x_3-y_3) \quad + \quad (x_1-y_1)(x_2-y_1) \quad + \quad (x_1-y_1)(x_1-y_2)$$

$$\mathfrak{S}_{2143}(x;y) = (x_1-y_1)(x_3-y_1) \quad + \quad (x_1-y_1)(x_2-y_2) \quad + \quad (x_1-y_1)(x_1-y_3)$$

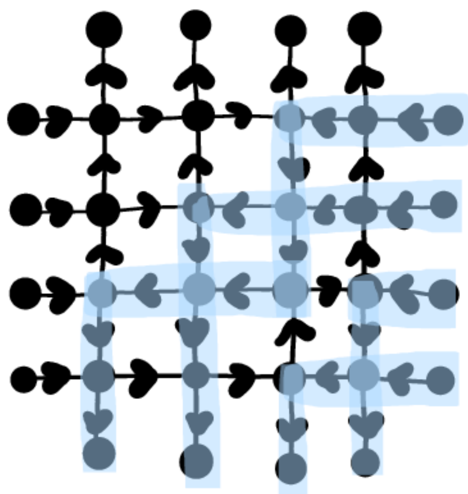


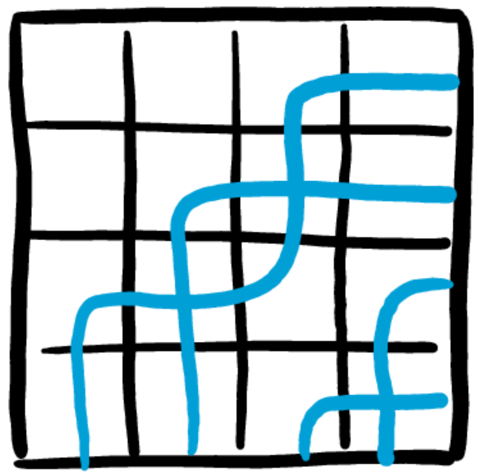
$$\mathfrak{S}_{2143}(x) = x_1x_3 \quad + \quad x_1x_2 \quad + \quad x_1^2$$

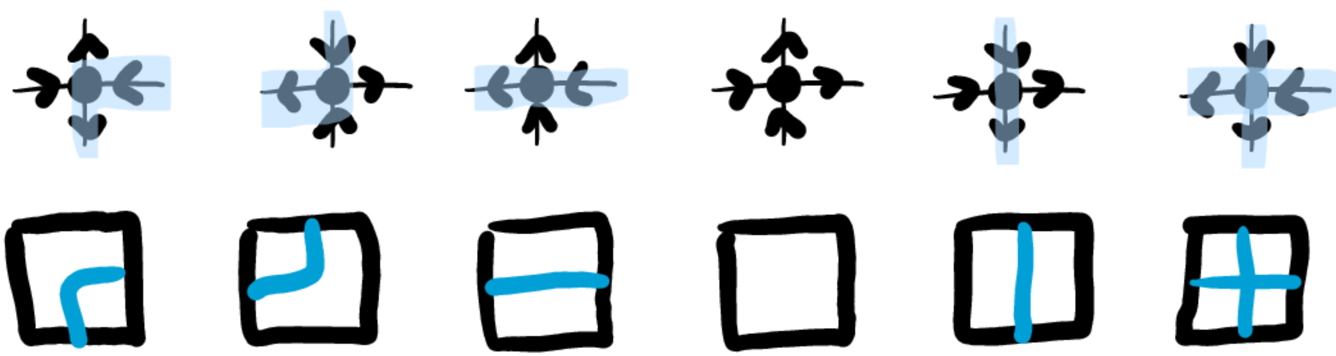


$$\mathfrak{S}_{2143}(x;y) = (x_1-y_1)(x_3-y_3) \quad + \quad (x_1-y_1)(x_2-y_1) \quad + \quad (x_1-y_1)(x_1-y_2)$$

Ice





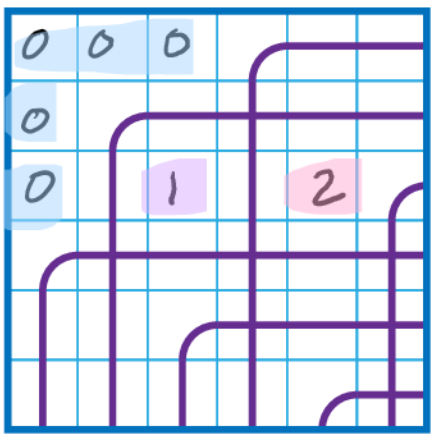


Lascoux: Chern and Yang Through Ice

Matrix Schubert Varieties

Schubert Determinantal Ideals

Given $w \in S_n$, the Schubert determinantal ideal $I_w \subseteq R = \mathbb{C}[z_{11}, z_{12}, \ldots, z_{nn}]$ is generated by minors of a generic matrix $Z = (z_{ij})_{i,j=1}^{n}$.



$I_{426135} = \langle z_{11}, z_{12}, z_{13}, z_{21}, z_{31} \rangle$

$+ \left\langle 2\times 2 \text{ minors in } \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} \right\rangle$

$+ \left\langle 3\times 3 \text{ minors in } \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} \end{bmatrix} \right\rangle$

Fulton's Generators

Matrix Schubert Varieties

$$\begin{tikzcd} GL(n) \arrow[r, hook, "i"] \arrow[d, two heads, "\pi"'] & Mat(n) \\ GL(n)/B & \end{tikzcd}$$

Flag variety

$$\begin{tikzcd} \pi^{-1}(X_w) \arrow[r, hook, "i"] \arrow[d, two heads] & \overline{i(\pi^{-1}(X_w))} \\ X_w & \end{tikzcd}$$

Matrix Schubert variety

Schubert variety

Write $\overline{X_w} := \overline{i(\pi^{-1}(X_w))}$.

Theorem (Fulton 1992): I_w is prime and

$\overline{X_w} = V(R/I_w)$.

Torus Equivariant Classes

Each $\overline{X_w}$ defines a class in the $T \times T$ equivariant cohomology of $Mat(n)$.

To compute $[X]_{T\times T}$ you can use three axioms

1. Degeneration
2. Additivity
3. Normalization.

Theorem (Feher-Rimanyi 2002, Knutson-Miller 2005):

$$[\overline{X_w}]_{T\times T} = \mathfrak{S}_w(x;y).$$

Corollary: $\mathfrak{S}_w(x)$ represents Schubert class in $H^*(GL(n)/B)$

A Recipe

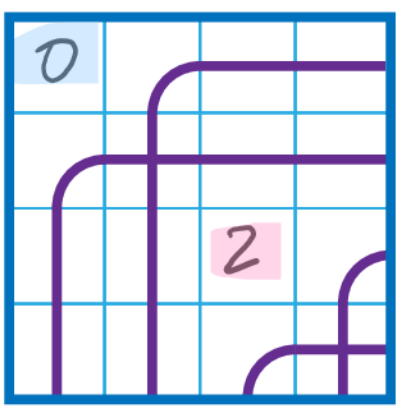
① Fix an antidiagonal term order $<_a$ on R.

② Compute the initial ideal $\mathrm{init}_{<_a}(I_w)$.

③ Take the primary decomposition.

Example:

$$I_{2143} = \left\langle z_{11},\ \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$



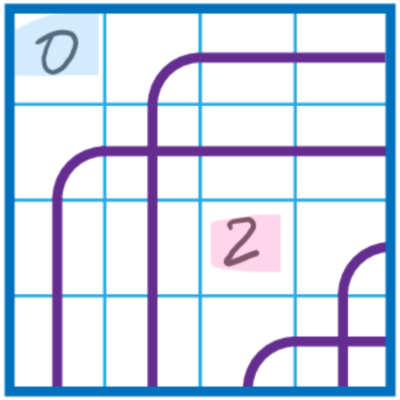
$$\mathrm{init}_{<_a}(I_{2143}) = \langle z_{11},\ z_{31} z_{22} z_{13} \rangle$$

$$= \langle z_{11}, z_{31} \rangle \cap \langle z_{11}, z_{22} \rangle \cap \langle z_{11}, z_{13} \rangle$$

$$\mathfrak{S}_{2143}(x;y) = [\overline{X_{2143}}]_{T\times T} = (x_1-y_1)(x_3-y_1) + (x_1-y_1)(x_2-y_2) + (x_1-y_1)(x_1-y_3)$$

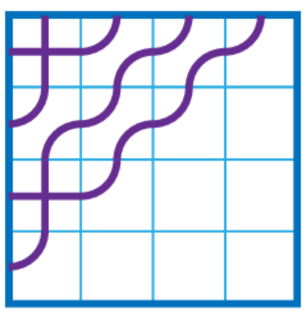
Example:

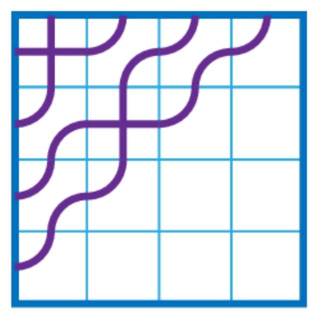
$$I_{2143} = \left\langle z_{11}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$

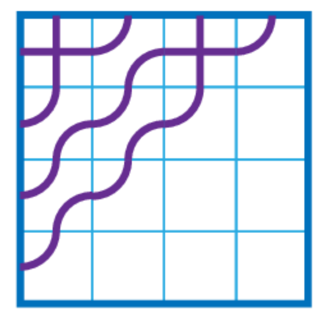


$$\mathrm{init}_{<a}(I_{2143}) = \langle z_{11}, z_{31} z_{22} z_{13} \rangle$$

$$= \langle z_{11}, z_{31} \rangle \cap \langle z_{11}, z_{22} \rangle \cap \langle z_{11}, z_{13} \rangle$$







$$\mathfrak{G}_{2143}(x;y) = (x_1-y_1)(x_3-y_1) + (x_1-y_1)(x_2-y_2) + (x_1-y_1)(x_1-y_3)$$

$$= [\overline{X}_{2143}]_{T \times T}$$

Theorem (Knutson - Miller 2005):

1. Fulton's generators are a Gröbner basis for I_w under any antidiagonal term order.

2. $\mathrm{init}_{<_a}(I_w)$ is radical.

3. $$\mathrm{init}_{<_a}(I_w) = \bigcap_{P \in \mathrm{Pipes}(w)} I_P$$

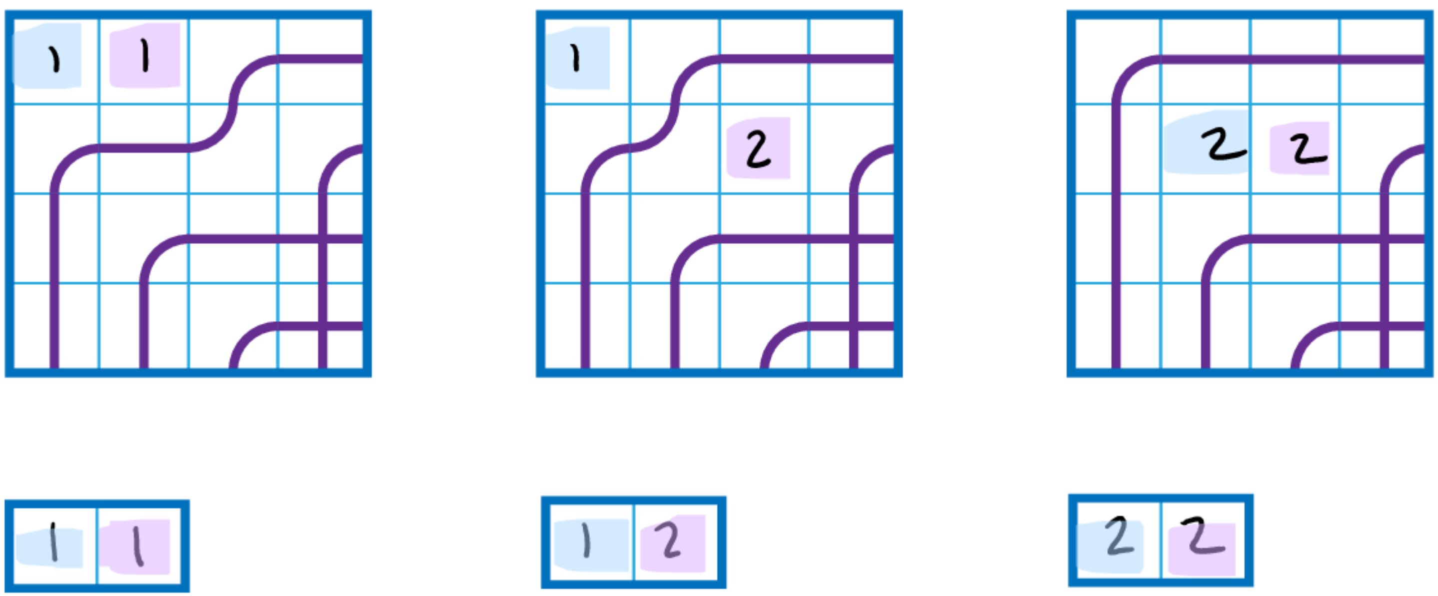
Diagonal Degenerations

Knutson, Miller, and Yong (2009) worked out an analagous story for diagonal degenerations of vexillary permutations.

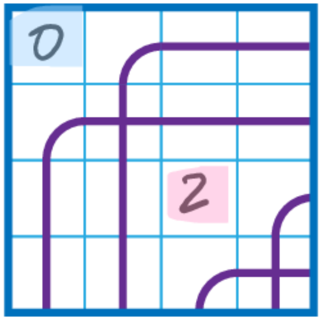
Theorem (KMY 2009):

1. Fulton's generators are a diagonal Gröbner basis if and only if w is vexillary.
2. In this case, primes are indexed by flagged tableaux.

Theorem (W-2020): Vexillary BPDs are in bijection with flagged tableaux



Example:

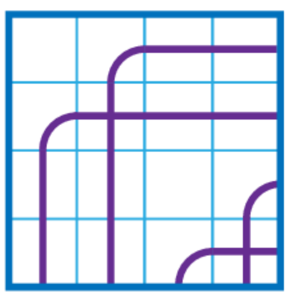


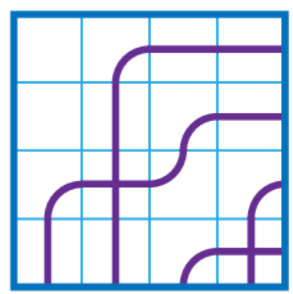
$$I_{2143} = \left\langle z_{11}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$

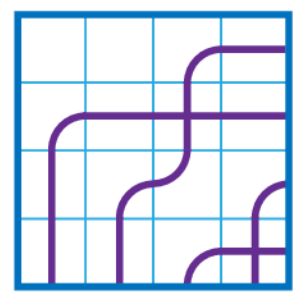
$$= \left\langle z_{11}, \begin{vmatrix} 0 & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$

$$\mathrm{init}_{<_d}(I_{2143}) = \langle z_{11}, z_{33} z_{21} z_{12} \rangle$$

$$= \langle z_{11}, z_{33} \rangle \cap \langle z_{11}, z_{21} \rangle \cap \langle z_{11}, z_{12} \rangle$$



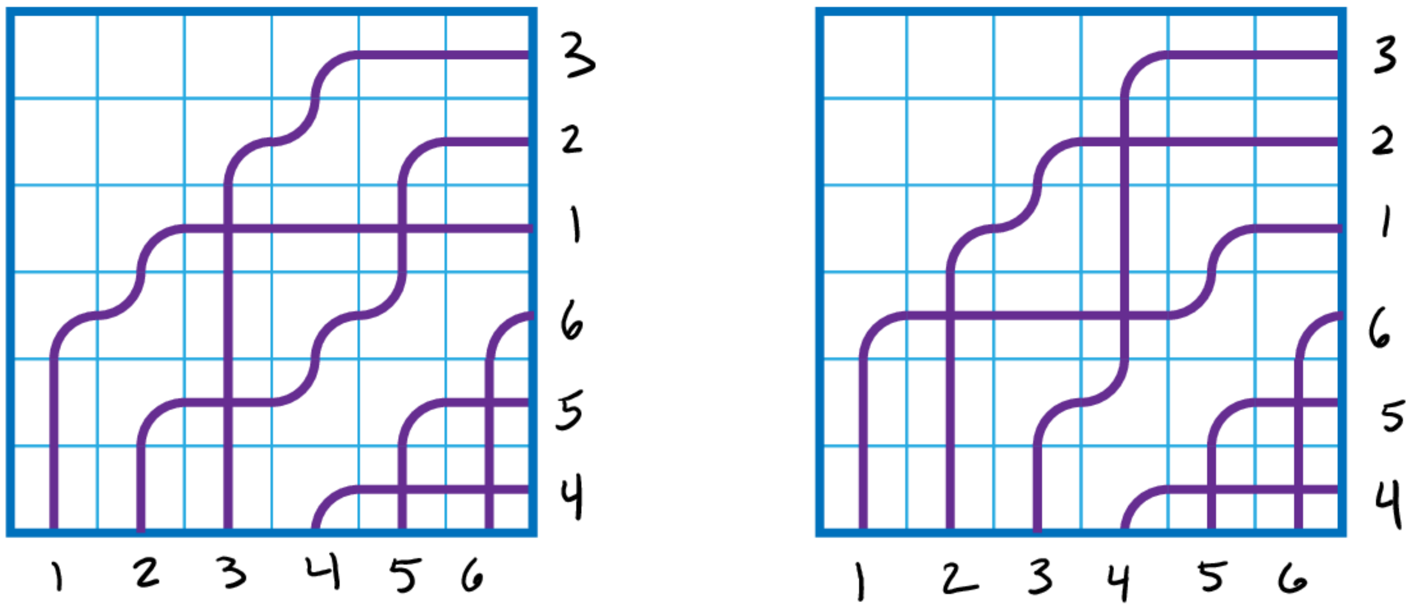




$$\mathfrak{G}_{2143}(x;y) = (x_1 - y_1)(x_3 - y_3) + (x_1 - y_1)(x_2 - y_1) + (x_1 - y_1)(x_1 - y_2)$$

A Pathology

Caution: $\mathrm{init}_{<_d}(I_w)$ might not be radical!



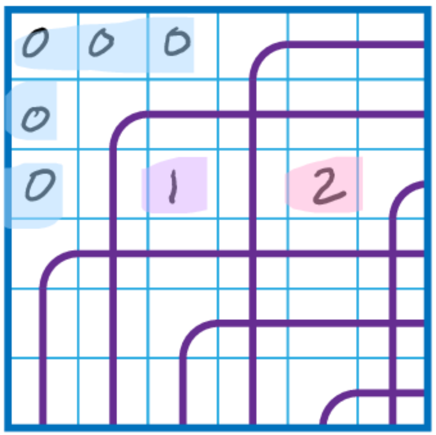
$V(R/\langle z_{11}, z_{12}, z_{13}, z_{21}, z_{22}, z_{31}\rangle)$ shows up in $\mathrm{Spec}(R/\mathrm{init}_{<_d}(I_{321654}))$ with multiplicity!

Conjecture (Hamaker-Pechenik-W. 2020):

BPD's for w label set theoretic components of $\mathrm{Spec}(R/\mathrm{init}_{\mathrm{ed}}(I_w))$ with multiplicity, i.e. the multiplicity of $V(R/\langle z_{ij} : (i,j) \in D \rangle)$ is

$$\#\{P \in BPD(w) : D \text{ indexes the blank tiles in } P\}.$$

CDG Generators



$$I_{426135} = \langle z_{11}, z_{12}, z_{13}, z_{21}, z_{31} \rangle$$

$$+ \left\langle \text{2x2 minors in } \begin{bmatrix} 0 & 0 & 0 \\ 0 & z_{22} & z_{23} \\ 0 & z_{32} & z_{33} \end{bmatrix} \right\rangle$$

$$+ \left\langle \text{3x3 minors in } \begin{bmatrix} 0 & 0 & 0 & z_{14} & z_{15} \\ 0 & z_{22} & z_{23} & z_{24} & z_{25} \\ 0 & z_{32} & z_{33} & z_{34} & z_{35} \end{bmatrix} \right\rangle$$

Question: When are CDG generators diagonal Gröbner bases?

Theorem (Hamaker–Pechenik–W. 2020):

If w is banner, then

1. The CDG generators are a diagonal Gröbner basis and

2. $$\mathrm{init}_{<_\alpha}(I_w) = \bigcap_{P \in BPD(w)} I_P.$$

Theorem (Klein 2020): The CDG generators for I_w are a diagonal Gröbner basis if and only if w avoids the patterns:

13254, 21543, 214635, 215364, 241635, 315264, 215634, and 4261735.

(Conjectured by Hamaker-Pechenik-W.)

Theorem (Klein–Weigandt 2021):

There is a diagonal term order σ so that the irreducible components of $\mathrm{Spec}(R/\mathrm{init}_\sigma(I_w))$ are labeled (with multiplicity) by elements of $BPD(w)$.

Thanks!